

THE BESSEL FUNCTION OF ORDER ZERO

Existence, uniqueness and an integral formula

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1 STATEMENT

DEFINITION 1.1 (*Bessel function*).

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{4^m (m!)^2} = 1 - \frac{z^2}{4} + \frac{z^4}{64} - \cdots. \quad (1.1)$$

THEOREM 1.2 (*Bessel's equation of order zero*). (a) J_0 is entire and satisfies

$$zf''(z) + f'(z) + zf(z) = 0 \quad (z \in \mathbb{C}). \quad (1.2)$$

(b) If f is analytic on a disc about 0 and satisfies (1.2) there, then $f = f(0)J_0$.

(c) For every $z \in \mathbb{C}$, $J_0(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta) d\theta$.

(d) $|J_0(x)| \leq 1$ for every real x .

2 SETTING

We use the following facts about power series [1, ch. 5]. A power series is analytic inside its disc of convergence and may be differentiated term by term there. If two power series agree on a disc about 0, they have the same coefficients.

3 IDEA OF THE PROOF

Substituting a general power series into (1.2) gives a two-step recursion $n^2 a_n = -a_{n-2}$, with $a_1 = 0$. That recursion has exactly one solution for each choice of a_0 , and $a_0 = 1$ gives the coefficients of J_0 .

4 PROOF

LEMMA 4.1 (*The recursion*). Let $f(z) = \sum_{n \geq 0} a_n z^n$ converge on a disc about 0. Then f satisfies (1.2) on that disc if and only if

$$a_1 = 0 \quad \text{and} \quad n^2 a_n + a_{n-2} = 0 \quad (n \geq 2).$$

PROOF. Differentiating term by term and collecting the coefficient of z^{n-1} ,

$$zf'' = \sum_{n \geq 2} n(n-1)a_n z^{n-1}, \quad f' = \sum_{n \geq 1} n a_n z^{n-1}, \quad zf = \sum_{n \geq 2} a_{n-2} z^{n-1}.$$

So $zf'' + f' + zf = a_1 + \sum_{n \geq 2} (n^2 a_n + a_{n-2})z^{n-1}$. This vanishes identically if and only if every coefficient is zero. ■

Proof of (a) and (b). Solve the recursion of Lemma 4.1. From $a_1 = 0$ and $a_{2m+1} = -a_{2m-1}/(2m+1)^2$, every odd coefficient is zero. For the even ones, $a_{2m} = -a_{2m-2}/(4m^2)$, so by induction

$$a_{2m} = \frac{(-1)^m a_0}{4^m (m!)^2}.$$

For (b): f has a power series at 0, which must satisfy the recursion, so $f = a_0 J_0$ with $a_0 = f(0)$.

For (a): the coefficients of J_0 satisfy the recursion with $a_0 = 1$, so it remains to see that (1.1) converges everywhere. For $z \neq 0$, the ratio of consecutive terms is $|z|^2/(4(m+1)^2) \rightarrow 0$, so by the ratio test the radius of convergence is infinite. Then J_0 is entire, and Lemma 4.1 applies on every disc. ■

LEMMA 4.2 (Wallis). For $m \geq 0$, $\int_0^\pi \sin^{2m} \theta \, d\theta = \pi \frac{(2m)!}{4^m (m!)^2}$.

PROOF. Let I_m be the integral, so $I_0 = \pi$. For $m \geq 1$, integrate by parts with $u = \sin^{2m-1} \theta$ and $dv = \sin \theta \, d\theta$. The boundary terms vanish, so

$$I_m = (2m-1) \int_0^\pi \sin^{2m-2} \theta \cos^2 \theta \, d\theta = (2m-1)(I_{m-1} - I_m),$$

that is, $I_m = \frac{2m-1}{2m} I_{m-1}$. The right-hand side of the lemma satisfies the same recursion, because

$$\frac{(2m)!}{4^m (m!)^2} \bigg/ \frac{(2m-2)!}{4^{m-1} ((m-1)!)^2} = \frac{(2m)(2m-1)}{4m^2} = \frac{2m-1}{2m},$$

and both sides equal π when $m = 0$. ■

Proof of (c). Fix z . The series $\cos(z \sin \theta) = \sum_m (-1)^m z^{2m} \sin^{2m} \theta / (2m)!$ converges uniformly in $\theta \in [0, \pi]$, since its m th term is at most $|z|^{2m} / (2m)!$. So we may integrate term by term and apply Lemma 4.2:

$$\frac{1}{\pi} \int_0^\pi \cos(z \sin \theta) \, d\theta = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{(2m)!} \cdot \frac{(2m)!}{4^m (m!)^2} = J_0(z). \quad \blacksquare$$

Proof of (d). For real x the integrand in (c) is a real number in $[-1, 1]$, so $|J_0(x)| \leq \frac{1}{\pi} \int_0^\pi 1 \, d\theta = 1$. ■

5 REMARKS

Remark 5.1 (An energy proof of (d)). Part (d) also follows from the equation alone. For real $x > 0$, put $E = J_0^2 + (J_0')^2$. By (1.2), $J_0'' = -J_0 - J_0'/x$, so

$$E' = 2J_0'(J_0 + J_0'') = -\frac{2(J_0')^2}{x} \leq 0.$$

Hence $E(x) \leq E(0) = 1$, so $|J_0(x)| \leq 1$. Since J_0 is even, the same holds for $x < 0$. Physically, the term f'/x acts as friction, so the oscillation loses energy and its amplitude decays.

Remark 5.2. Away from 0, (1.2) is a second-order linear equation, so it has a two-dimensional space of solutions on any disc not containing 0. By (b), only the multiples of J_0 extend analytically to 0. Every independent second solution is singular at the origin; the standard choice is the Bessel function Y_0 of the second kind, which has a logarithmic singularity there.

REFERENCES

- [1] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering and Science*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2003.