

ENTIRE FUNCTIONS OF POLYNOMIAL GROWTH

Liouville's theorem, one derivative and one power at a time

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I STATEMENT

THEOREM 1.1 (*Polynomial growth*). Let f be entire and $k \geq 0$ an integer. The following are equivalent:

- (a) f is a polynomial of degree at most k ;
- (b) $f^{(k)}$ is bounded on $\mathbb{C}$;
- (c) there are constants $A, B \geq 0$ with $|f(z)| \leq A + B|z|^k$ for all $z \in \mathbb{C}$.

The case $k = 0$ of (c) $\Rightarrow$ (a) is Liouville's theorem.

2 SETTING

We use two standard facts about an entire function f [I, ch. 4–5]: it equals its Taylor series $f(z) = \sum_{n \geq 0} a_n z^n$ on all of $\mathbb{C}$, with $a_n = f^{(n)}(0)/n!$, and its derivatives are again entire.

LEMMA 2.1 (*Cauchy's estimate*). If f is entire and $M(R) = \max_{|z|=R} |f(z)|$, then $|a_n| \leq M(R)/R^n$ for every $R > 0$ and $n \geq 0$.

PROOF. By Cauchy's integral formula for derivatives on the circle $|z| = R$,

$$|a_n| = \left| \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{M(R)}{R^{n+1}} = \frac{M(R)}{R^n}. \quad \blacksquare$$

3 IDEA OF THE PROOF

Growth of f controls its Taylor coefficients through **Lemma 2.1**. If f grows no faster than $|z|^k$, every coefficient beyond a_k is crushed to zero as $R \rightarrow \infty$.

4 PROOF

(c) $\Rightarrow$ (a). For $R \geq 1$ we have $M(R) \leq A + BR^k$, so for $n > k$, **Lemma 2.1** gives

$$|a_n| \leq \frac{A + BR^k}{R^n} = \frac{A}{R^n} + \frac{B}{R^{n-k}} \xrightarrow{R \rightarrow \infty} 0.$$

Hence $a_n = 0$ for all $n > k$, and $f(z) = \sum_{n=0}^k a_n z^n$. ■

(b) $\Rightarrow$ (a). The function $f^{(k)}$ is entire and bounded, so it satisfies (c) with exponent 0. By the case already proved, it is a constant c . Then $f^{(n)} \equiv 0$ for $n > k$, so $a_n = f^{(n)}(0)/n! = 0$ for $n > k$. ■

(a) $\Rightarrow$ (b) and (c). If $f(z) = \sum_{n=0}^k a_n z^n$, then $f^{(k)} = k! a_k$ is constant. For (c), note that $|z|^n \leq 1 + |z|^k$ for $0 \leq n \leq k$, since one of 1 and $|z|^k$ is at least $|z|^n$. So $|f(z)| \leq \sum_n |a_n| (1 + |z|^k)$. ■

5 REMARKS

COROLLARY 5.1 (*Sublinear growth*). *If f is entire and $f(z)/z \rightarrow 0$ as $z \rightarrow \infty$, then f is constant.*

PROOF. By Lemma 2.1 with $n = 1$, $|a_1| \leq M(R)/R \rightarrow 0$, and for $n \geq 2$, $|a_n| \leq M(R)/R^n \leq M(R)/R \rightarrow 0$ once $R \geq 1$. ■

Remark 5.2. Nothing like this holds for real functions. The function $\sin x$ is smooth on $\mathbb{R}$ and all of its derivatives are bounded by 1, but it is not a polynomial. Its complex extension is unbounded in every direction except the real axis: $|\sin(iy)| = |\sinh y|$. Bounding a function on the real line says nothing about the complex plane, and Liouville's theorem needs the whole plane.

Remark 5.3. Read contrapositively, (c) $\Rightarrow$ (a) says that an entire function that is not a polynomial, such as e^z , outgrows every power: for each k there is a sequence $z_j \rightarrow \infty$ with $|f(z_j)|/|z_j|^k \rightarrow \infty$.

REFERENCES

- [1] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering and Science*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2003.