

THE REFLECTIVE PROPERTY OF THE ELLIPSE

Two coordinate-free proofs

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1 STATEMENT

THEOREM 1.1 (Reflective property). *Let E be an ellipse with foci F_1, F_2 . At every point $P \in E$, the normal line to E bisects the angle F_1PF_2 . Consequently, a ray from F_1 that reflects off E at P passes through F_2 .*

2 SETTING

DEFINITION 2.1 (Ellipse). Let $F_1, F_2 \in \mathbb{R}^2$ and $a > c \geq 0$, where $2c = |F_1 - F_2|$. The ellipse with foci F_1, F_2 and major axis $2a$ is the level set

$$E = \{X \in \mathbb{R}^2 : g(X) = 2a\}, \quad g(X) = |X - F_1| + |X - F_2|.$$

DEFINITION 2.2 (Law of reflection). A ray travelling in the unit direction d that meets a curve with unit normal n leaves in the direction $d' = d - 2(d \cdot n)n$, the mirror image of d in the tangent line.

The circle is the case $c = 0$, and no point needs separate treatment: unlike a proof using slopes, nothing goes wrong where the tangent or a focal line is vertical.

3 IDEA OF THE PROOF

The gradient of a distance $|X - F|$ is the unit vector pointing away from F . So ∇g is a sum of two unit vectors, and a sum of two unit vectors bisects the angle between them.

4 FIRST PROOF: THE GRADIENT

LEMMA 4.1 (E is a smooth curve). *Neither focus lies on E . At every $X \in E$, g is differentiable with*

$$\nabla g(X) = u_1 + u_2 \neq 0, \quad u_j = \frac{X - F_j}{|X - F_j|}.$$

PROOF. By the triangle inequality, $g(F_1) = g(F_2) = 2c < 2a$, so the foci are not on E . Away from F_j , the map $X \mapsto |X - F_j|$ is differentiable with gradient u_j . If $u_1 + u_2 = 0$, then X lies on the segment F_1F_2 , so $g(X) = 2c < 2a$ and $X \notin E$. ■

By Lemma 4.1 and the implicit function theorem, E is a smooth curve near each of its points, with normal direction $\nabla g = u_1 + u_2$.

Proof of Theorem 1.1. Let $P \in E$ and set u_j as in Lemma 4.1, so u_1 points from F_1 to P and u_2 from F_2 to P . Since $|u_1| = |u_2| = 1$,

$$u_1 \cdot (u_1 + u_2) = 1 + u_1 \cdot u_2 = u_2 \cdot (u_1 + u_2),$$

so u_1 and u_2 make equal angles with the normal $u_1 + u_2$. Equivalently, $-u_1$ and $-u_2$, which point from P to the foci, make equal angles with the inward normal. This is the first claim.

For the second, the incoming ray travels in direction $d = u_1$. Put $s = 1 + u_1 \cdot u_2 > 0$, so that $|u_1 + u_2|^2 = 2s$ and the unit normal is $n = (u_1 + u_2)/\sqrt{2s}$. Then $d \cdot n = s/\sqrt{2s}$ and

$$d' = u_1 - 2 \cdot \frac{s}{\sqrt{2s}} \cdot \frac{u_1 + u_2}{\sqrt{2s}} = u_1 - (u_1 + u_2) = -u_2,$$

which is the direction from P towards F_2 . ■

5 SECOND PROOF: HERON'S SHORTEST PATH

This proof needs no calculus, only the triangle inequality.

LEMMA 5.1 (Supporting line). Let ℓ be the tangent line to E at P . Then $g(X) > 2a$ for every $X \in \ell$ with $X \neq P$.

PROOF. The function g is convex, as a sum of two convex functions. Restricted to ℓ , it is a convex function of one variable whose derivative at P is $\nabla g(P) \cdot v = 0$ for a direction v of ℓ . So P minimises g on ℓ , and $g \geq 2a$ there. If $g(X) = 2a$ for some $X \neq P$ on ℓ , then by convexity $g = 2a$ on the whole segment PX . Each $|Y - F_j|$ is convex, so each must then be affine along PX . A distance $|Y - F_j|$ is affine along a segment only if the line through it passes through F_j , so ℓ would contain both foci. But $g(F_1) = 2c < 2a$, contradicting $g \geq 2a$ on ℓ . ■

Second proof of Theorem 1.1. Let F'_2 be the reflection of F_2 in ℓ , so that $|X - F_2| = |X - F'_2|$ for all $X \in \ell$. Both foci satisfy $g < 2a$, and the set $\{g < 2a\}$ is convex and does not meet ℓ by Lemma 5.1, so F_1 and F_2 lie on the same side of ℓ . Hence F_1 and F'_2 lie on opposite sides, and the segment $F_1F'_2$ meets ℓ at a point X_0 .

For any $X \in \ell$, the triangle inequality gives

$$g(X) = |X - F_1| + |X - F'_2| \geq |F_1 - F'_2| = g(X_0).$$

So X_0 minimises g on ℓ , and by Lemma 5.1 the only minimiser is P . Therefore P lies on the segment $F_1F'_2$. The line F_1P continues straight to F'_2 , and reflecting that continuation in ℓ sends it to PF_2 . This is exactly the law of reflection at P . ■

6 REMARKS

Remark 6.1. The same computation proves the hyperbola's property. With $b(X) = |X - F_1| - |X - F_2|$ we have $\nabla b = u_1 - u_2$, and

$$u_1 \cdot (u_1 - u_2) = 1 - u_1 \cdot u_2 = -u_2 \cdot (u_1 - u_2).$$

So the *tangent* bisects the angle between the focal lines. A ray aimed at one focus is reflected towards the other.

Remark 6.2. The Analytic Geometry notes prove the theorem with slopes and the tangent-of-angle formula. That needs a limiting argument at the four points where a focal line is vertical. The gradient proof treats every point the same way.