

EULER'S IDENTITY

Two proofs: power series and the circle group

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1 STATEMENT

THEOREM 1.1 (*Euler's identity*).

$$e^{i\pi} + 1 = 0.$$

2 SETTING

Everything is built from power series, so no geometric intuition about angles is assumed. Section 5 gives a second proof using group theory.

DEFINITION 2.1 (*Exponential, sine and cosine*). For $z \in \mathbb{C}$, define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \quad \sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad (2.1)$$

and write $e^z = \exp(z)$.

DEFINITION 2.2 (*The number π*). π is twice the smallest positive real zero of $\cos$.

3 IDEA OF THE PROOF

Substitute $i\theta$ into the series for $\exp$. The powers of i cycle through $1, i, -1, -i$, so the even terms rebuild $\cos \theta$ and the odd terms rebuild $i \sin \theta$. Then evaluate at $\theta = \pi$.

4 PROOF

LEMMA 4.1 (*Absolute convergence*). Each series in (2.1) converges absolutely for every $z \in \mathbb{C}$.

PROOF. For $\exp$, the ratio of consecutive terms of $\sum |z|^n/n!$ is $|z|/(n+1) \rightarrow 0$, so the series converges by the ratio test. The series for $\cos$ and $\sin$ are bounded term by term by this one. ■

LEMMA 4.2 (*Euler's formula*). For every $\theta \in \mathbb{R}$, $e^{i\theta} = \cos \theta + i \sin \theta$.

PROOF. Since $i^{2k} = (-1)^k$ and $i^{2k+1} = (-1)^k i$, by Lemma 4.1 we may split the series into even and odd terms:

$$\begin{aligned} e^{i\theta} &= \sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} = \cos \theta + i \sin \theta. \end{aligned} \quad \blacksquare$$

LEMMA 4.3 (Values at π). $\cos \pi = -1$ and $\sin \pi = 0$.

PROOF. Differentiating (2.1) term by term gives $\sin' = \cos$ and $\cos' = -\sin$, so $\frac{d}{dx}(\sin^2 x + \cos^2 x) = 0$ and $\sin^2 + \cos^2 = 1$.

By Definition 2.2, $\cos(\pi/2) = 0$, so $\sin(\pi/2) = \pm 1$. Since $\sin' = \cos > 0$ on $[0, \pi/2)$ and $\sin 0 = 0$, $\sin(\pi/2) = 1$. The addition formulas, which follow from $e^{a+b} = e^a e^b$ and Lemma 4.2, then give

$$\cos \pi = \cos^2 \frac{\pi}{2} - \sin^2 \frac{\pi}{2} = -1, \quad \sin \pi = 2 \sin \frac{\pi}{2} \cos \frac{\pi}{2} = 0. \quad \blacksquare$$

Proof of Theorem 1.1. By Lemmas 4.2 and 4.3,

$$e^{i\pi} = \cos \pi + i \sin \pi = -1,$$

so $e^{i\pi} + 1 = 0$. $\blacksquare$

5 A SECOND PROOF: THE CIRCLE GROUP

The first proof computes $\cos \pi$ and $\sin \pi$ through the addition formulas. The second avoids them: it needs only the value at $\pi/2$ and the fact that $\exp$ turns addition into multiplication.

LEMMA 5.1 (Homomorphism). $\exp(z + w) = \exp(z) \exp(w)$ for all $z, w \in \mathbb{C}$, and $\exp(0) = 1$.

PROOF. The series for $\exp(z)$ and $\exp(w)$ converge absolutely by Lemma 4.1, so their Cauchy product converges to $\exp(z) \exp(w)$. Its n th term is

$$\sum_{k=0}^n \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} z^k w^{n-k} = \frac{(z+w)^n}{n!}$$

by the binomial theorem, which is the n th term of $\exp(z+w)$. $\blacksquare$

Let $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$. It is a group under multiplication, called the *circle group*.

PROPOSITION 5.2 (Into the circle). The map $E(\theta) = e^{i\theta}$ is a group homomorphism $E: (\mathbb{R}, +) \rightarrow (\mathbb{T}, \cdot)$.

PROOF. The series for $\exp$ has real coefficients, so $\overline{e^{i\theta}} = e^{-i\theta}$. By Lemma 5.1,

$$|e^{i\theta}|^2 = e^{i\theta} e^{-i\theta} = e^0 = 1,$$

so E takes values in $\mathbb{T}$. It is a homomorphism by Lemma 5.1. $\blacksquare$

LEMMA 5.3 (A quarter turn). $E(\pi/2) = i$.

PROOF. By Lemma 4.2 and Definition 2.2, $E(\pi/2) = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i \sin \frac{\pi}{2}$. This has modulus 1 by Proposition 5.2, so $\sin \frac{\pi}{2} = \pm 1$. On $[0, \pi/2)$ we have $\sin' = \cos > 0$ and $\sin 0 = 0$, so $\sin \frac{\pi}{2} > 0$ and $E(\pi/2) = i$. ■

Second proof of Theorem 1.1. E is a homomorphism, so

$$e^{i\pi} = E\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = E\left(\frac{\pi}{2}\right)^2 = i^2 = -1. \quad \blacksquare$$

Remark 5.4 (The group-theoretic meaning). More is true [2, Prologue]: E is surjective with kernel $2\pi\mathbb{Z}$, so it induces an isomorphism of groups

$$\mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{T}, \quad \theta + 2\pi\mathbb{Z} \mapsto e^{i\theta}.$$

Each side has exactly one element of order 2: the class of π on the left, and -1 on the right, since $w^2 = 1$ forces $w = \pm 1$. An isomorphism must match them. In this language, Euler's identity says that half a turn is -1 .

6 REMARKS

Remark 6.1. The identity links five constants $(0, 1, e, i, \pi)$ using addition, multiplication and exponentiation exactly once each. The proof shows it is less mysterious than it looks: it is Lemma 4.2 at a single point.

Remark 6.2. Two facts are used without proof here: that power series can be differentiated term by term inside their radius of convergence, and that $\cos$ has a positive zero (true since $\cos 0 = 1$ and $\cos 2 < 0$ from the alternating series).

REFERENCES

- [1] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw–Hill, 1976, ch. 8.
- [2] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw–Hill, New York, 1987.