

THE GOLDEN RATIO IS IRRATIONAL

Three proofs: descent, rational roots and continued fractions

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1 STATEMENT

THEOREM 1.1 (*Irrationality of φ*). The golden ratio $\varphi = \frac{1 + \sqrt{5}}{2}$ is irrational.

2 SETTING

The number φ is the positive root of

$$x^2 - x - 1 = 0, \quad \text{equivalently} \quad \varphi = 1 + \frac{1}{\varphi}. \quad (2.1)$$

Since $2 < \sqrt{5} < 3$, we have $1 < \varphi < 2$.

3 IDEA OF THE PROOF

The relation $\varphi = 1 + 1/\varphi$ is self-similar: removing a unit from φ and inverting gives φ back. A fraction cannot behave like this, because each step strictly shrinks its denominator.

4 FIRST PROOF: DESCENT

PROOF. Suppose φ is rational. Among all ways of writing $\varphi = p/q$ with integers $p, q \geq 1$, choose one with q least, by well-ordering. By (2.1),

$$\varphi = \frac{1}{\varphi - 1} = \frac{1}{p/q - 1} = \frac{q}{p - q}.$$

Since $1 < p/q < 2$, we have $0 < p - q < q$. This writes φ with the positive denominator $p - q < q$, contradicting the choice of q . ■

5 SECOND PROOF: RATIONAL ROOTS

LEMMA 5.1 (*Rational root test*). If p/q is in lowest terms and is a root of $ax^2 + bx + c$ with integer coefficients, then $p \mid c$ and $q \mid a$.

PROOF. Multiplying $a(p/q)^2 + b(p/q) + c = 0$ by q^2 gives $ap^2 + bpq + cq^2 = 0$. So $p \mid cq^2$. Since $\gcd(p, q) = 1$, also $\gcd(p, q^2) = 1$, so Bézout gives $1 = px + q^2y$ and $c = pcx + cq^2y$ is divisible by p . Similarly, $q \mid ap^2$ gives $q \mid a$. ■

Second proof of Theorem 1.1. By Lemma 5.1, a rational root of $x^2 - x - 1$ has $p \mid 1$ and $q \mid 1$, so it is ± 1 . But $1 - 1 - 1 = -1$ and $1 + 1 - 1 = 1$, so neither is a root. ■

6 THIRD PROOF: CONTINUED FRACTIONS

For $x > 0$, the *continued fraction algorithm* sets $x_0 = x$, records $a_j = \lfloor x_j \rfloor$, and continues with $x_{j+1} = 1/(x_j - a_j)$, stopping if x_j is an integer.

LEMMA 6.1 (*Rationals terminate*). *If x is rational, the algorithm stops after finitely many steps.*

PROOF. If $x_j = p/q$ with $q \geq 1$ is not an integer, write $p = a_j q + r$ with $0 < r < q$. Then $x_{j+1} = q/r$ has denominator $r < q$. Denominators are positive integers that strictly decrease, so the process stops. ■

Third proof of Theorem 1.1. For $x_0 = \varphi$: $a_0 = 1$ because $1 < \varphi < 2$, and $x_1 = 1/(\varphi - 1) = \varphi$ by (2.1). So $x_j = \varphi$ for every j , and the algorithm never stops:

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$$

By Lemma 6.1, φ is irrational. ■

7 REMARKS

Remark 7.1. The three proofs are one proof in different clothing. The step $p/q \mapsto q/(p - q)$ in the first proof is one step of the continued fraction algorithm, and it is the Euclidean algorithm run on (p, q) .

Remark 7.2. Truncating the continued fraction gives $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \dots$, the ratios F_{j+1}/F_j of consecutive Fibonacci numbers. If $r_j = F_{j+1}/F_j$, then $r_{j+1} = 1 + 1/r_j$, the same recursion as (2.1). One can show $|r_j - \varphi| < 1/F_j^2$, so these fractions approximate φ , but only slowly: every partial quotient is 1, the smallest possible.

Remark 7.3. This sharpens the treatment in the Foundations notes, which reduces the problem to $\sqrt{5}$ and residues mod 5. Here no fact about $\sqrt{5}$ is needed beyond $2 < \sqrt{5} < 3$.