

A POWER SERIES WITH A NATURAL BOUNDARY

Why $z + z^2 + z^4 + z^8 + \dots$ cannot be continued past the unit circle

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1 STATEMENT

Let $\mathbb{D} = \{z : |z| < 1\}$ and $\mathbb{T} = \{z : |z| = 1\}$, and

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \dots$$

THEOREM 1.1 (*Natural boundary*). The series f has radius of convergence 1, and every point of $\mathbb{T}$ is a singular point of f . So f cannot be continued analytically across any arc of the unit circle.

2 SETTING

DEFINITION 2.1 (*Regular and singular points*). A point $\zeta \in \mathbb{T}$ is *regular* for f if there are an open disc D centred at ζ and a function g analytic on D with $g = f$ on $D \cap \mathbb{D}$. Otherwise ζ is *singular*. If every point of $\mathbb{T}$ is singular, $\mathbb{T}$ is a *natural boundary* for f .

This is the standard notion [1, ch. 16]. A power series with radius of convergence 1 always has at least one singular point on $\mathbb{T}$; for instance $1/(1-z)$ has exactly one. The theorem says that f is as bad as possible.

DEFINITION 2.2 (*Dyadic roots of unity*). $\omega \in \mathbb{T}$ is a *dyadic root of unity* if $\omega^{2^m} = 1$ for some $m \geq 0$, that is, $\omega = e^{2\pi i j / 2^m}$ for some integers j and $m \geq 0$.

3 IDEA OF THE PROOF

Along the radius towards a dyadic root of unity ω , all but finitely many terms $(r\omega)^{2^n}$ equal r^{2^n} , which is positive. So f blows up along that radius, and ω cannot be regular. Such roots are dense in the circle, and the regular points form an open set, so there are no regular points at all.

4 PROOF

LEMMA 4.1 (*Radius of convergence*). The radius of convergence of f is 1.

PROOF. For $|z| < 1$ the series is dominated by $\sum_k |z|^k < \infty$. At $z = 1$ its terms do not tend to 0, so it diverges, and the radius is exactly 1. ■

LEMMA 4.2 (*Blow-up at dyadic roots*). If $\omega^{2^m} = 1$, then $|f(r\omega)| \rightarrow \infty$ as $r \rightarrow 1^-$.

PROOF. For $n \geq m$, $\omega^{2^n} = (\omega^{2^m})^{2^{n-m}} = 1$. Hence, for $0 < r < 1$,

$$f(r\omega) = \sum_{n=0}^{m-1} (r\omega)^{2^n} + \sum_{n=m}^{\infty} r^{2^n}.$$

The first sum has modulus at most m . For any $N > m$, the second sum is at least $\sum_{n=m}^N r^{2^n}$, which tends to $N - m + 1$ as $r \rightarrow 1^-$. So

$$\liminf_{r \rightarrow 1^-} |f(r\omega)| \geq (N - m + 1) - m$$

for every N , and the limit is infinite. ■

LEMMA 4.3 (Density). *The dyadic roots of unity are dense in $\mathbb{T}$.*

PROOF. Given $e^{i\theta}$ and $m \geq 0$, choose an integer j with $|\theta - 2\pi j/2^m| \leq \pi/2^m$. Since $|e^{is} - e^{it}| \leq |s - t|$, the root $e^{2\pi i j/2^m}$ lies within $\pi/2^m$ of $e^{i\theta}$. ■

LEMMA 4.4 (Regular points form an open set). *If ζ is regular, then so is every point of $\mathbb{T}$ close enough to ζ .*

PROOF. Let D and g be as in Definition 2.1. For $\zeta' \in \mathbb{T} \cap D$, choose an open disc D' centred at ζ' with $D' \subseteq D$. Then g is analytic on D' and equals f on $D' \cap \mathbb{D}$. ■

Proof of Theorem 1.1. A dyadic root ω is singular. If it were regular, with g analytic on a disc D about ω , then g would be bounded near ω . But $f(r\omega) = g(r\omega)$ for r close to 1, which is unbounded by Lemma 4.2.

By Lemma 4.4, the set of regular points is an open subset of $\mathbb{T}$, and it contains no dyadic root. By Lemma 4.3, every non-empty open subset of $\mathbb{T}$ contains a dyadic root. Hence the set of regular points is empty. ■

5 REMARKS

Remark 5.1. The functional equation $f(z) = z + f(z^2)$ explains the result. Singularities of $f(z^2)$ include every square root of a singularity of f , so once 1 is singular, so are -1 , then $\pm i$, and so on through all dyadic roots of unity.

Remark 5.2. The real reason is the size of the gaps between exponents. By Hadamard's gap theorem, any power series $\sum c_k z^{n_k}$ with radius of convergence 1 and $n_{k+1}/n_k \geq \lambda > 1$ for all k has $\mathbb{T}$ as a natural boundary. Our proof uses the special structure $n_k = 2^k$ to avoid that theorem.

REFERENCES

- [1] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.