

THREE TANGENTS TO A PARABOLA

A ratio identity, for every parabola and every third tangent

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1 STATEMENT

Let T_1 and T_2 be tangents to a parabola at distinct points P_1 and P_2 , meeting at P . A third tangent T , at a point other than P_1 and P_2 , meets T_1 at Q_1 and T_2 at Q_2 . Since Q_1 lies on the line PP_1 , there is a unique real number t_1 with $Q_1 = P + t_1(P_1 - P)$; define t_2 in the same way from $Q_2 = P + t_2(P_2 - P)$. Thus $|t_1| = PQ_1/PP_1$ and $|t_2| = PQ_2/PP_2$.

THEOREM 1.1 (*Three tangents*). *In this situation $t_1 + t_2 = 1$. If the point of contact of T lies on the arc between P_1 and P_2 , then $0 < t_1, t_2 < 1$, and hence*

$$\frac{PQ_1}{PP_1} + \frac{PQ_2}{PP_2} = 1.$$

2 SETTING

DEFINITION 2.1 (*Similarity*). A *similarity* of the plane is a map $S(X) = \lambda RX + v$, where $\lambda > 0$, R is a rotation or reflection, and $v \in \mathbb{R}^2$.

A similarity is an affine bijection. It maps lines to lines, preserves tangency between lines and smooth curves, and multiplies every length by λ . In particular, it preserves the coefficients t_1 and t_2 , because $S(P + t(P_1 - P)) = S(P) + t(S(P_1) - S(P))$.

LEMMA 2.2 (*All parabolas are similar*). *Every parabola is the image of $y = x^2$ under a similarity.*

PROOF. A parabola with focal distance $p > 0$ can be moved by a rotation and a translation to $y = x^2/(4p)$. The scaling $(x, y) \mapsto (4p x, 4p y)$ takes $y = x^2$ to $y/(4p) = (x/(4p))^2$, that is, to $y = x^2/(4p)$. ■

By **Lemma 2.2** and the invariance of t_1 and t_2 , it suffices to prove **Theorem 1.1** for $y = x^2$.

3 IDEA OF THE PROOF

On $y = x^2$ the tangents at s and u meet at $(\frac{s+u}{2}, su)$. Measured along a tangent line, the parameters t_1 and t_2 are ratios of x -coordinates, and they add to one by a single cancellation.

4 PROOF

LEMMA 4.1 (*Tangent intersections*). On $y = x^2$, the tangent at (s, s^2) is $y = 2sx - s^2$. For $s \neq u$, the tangents at s and u meet at $(\frac{s+u}{2}, su)$.

PROOF. The slope at $x = s$ is $2s$, so the tangent is $y - s^2 = 2s(x - s)$. Two such lines meet where $2(s - u)x = s^2 - u^2$, that is, at $x = \frac{s+u}{2}$, and then $y = 2s \cdot \frac{s+u}{2} - s^2 = su$. ■

Proof of Theorem 1.1. Let the three points of contact have x -coordinates a, b, c , all distinct, with $P_1 = (a, a^2)$ and $P_2 = (b, b^2)$. By Lemma 4.1,

$$P = \left(\frac{a+b}{2}, ab\right), \quad Q_1 = \left(\frac{a+c}{2}, ac\right), \quad Q_2 = \left(\frac{b+c}{2}, bc\right).$$

The line T_1 is not vertical, so t_1 is determined by x -coordinates alone:

$$t_1 = \frac{x_{Q_1} - x_P}{x_{P_1} - x_P} = \frac{\frac{a+c}{2} - \frac{a+b}{2}}{a - \frac{a+b}{2}} = \frac{c-b}{a-b}.$$

In the same way, on T_2 ,

$$t_2 = \frac{\frac{b+c}{2} - \frac{a+b}{2}}{b - \frac{a+b}{2}} = \frac{c-a}{b-a}.$$

Hence

$$t_1 + t_2 = \frac{c-b}{a-b} - \frac{c-a}{a-b} = \frac{a-b}{a-b} = 1.$$

If c lies strictly between a and b , then $c - b$ has the sign of $a - b$ and $|c - b| < |a - b|$, so $0 < t_1 < 1$; likewise $0 < t_2 < 1$. Then $PQ_1/PP_1 + PQ_2/PP_2 = t_1 + t_2 = 1$. ■

5 REMARKS

Remark 5.1. When the third point of contact lies outside the arc P_1P_2 , exactly one of t_1, t_2 is negative, and the identity becomes $\left|\frac{PQ_1}{PP_1} - \frac{PQ_2}{PP_2}\right| = 1$. The signed form $t_1 + t_2 = 1$ covers both cases.

Remark 5.2. Since $t_1 = \frac{c-b}{a-b}$ depends on c linearly, the map $Q_1 \mapsto Q_2$ from T_1 to T_2 is affine. A moving tangent therefore cuts two fixed tangents in proportional segments. This is the classical construction of a parabola as the envelope of lines joining corresponding points of two proportionally divided segments, as in string art.