

VARIGNON'S THEOREM

The midpoint parallelogram of any quadrilateral

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I STATEMENT

THEOREM 1.1 (Varignon). *Let A, B, C, D be any four points in $\mathbb{R}^n$, and let K, L, M, N be the midpoints of AB, BC, CD, DA . Then:*

- (a) $\overrightarrow{KL} = \overrightarrow{NM} = \frac{1}{2}\overrightarrow{AC}$ and $\overrightarrow{KN} = \overrightarrow{LM} = \frac{1}{2}\overrightarrow{BD}$, so $KLMN$ is a parallelogram, possibly degenerate;
- (b) its diagonals bisect each other at the centroid $G = \frac{1}{4}(A + B + C + D)$;
- (c) its perimeter is $AC + BD$;
- (d) if $ABCD$ is a simple quadrilateral in the plane, the area of $KLMN$ is half the area of $ABCD$.

The quadrilateral may be convex, non-convex or self-intersecting, and in (a)–(c) the four points need not even lie in a plane.

2 SETTING

We identify each point with its position vector, so that $\overrightarrow{PQ} = Q - P$ and the midpoint of PQ is $\frac{1}{2}(P + Q)$. A parallelogram is a quadrilateral $KLMN$ with $L - K = M - N$; equivalently, KL and NM are equal and parallel.

For vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2$, write

$$u \wedge v = u_1v_2 - u_2v_1.$$

This is bilinear and antisymmetric ($v \wedge u = -u \wedge v$, so $u \wedge u = 0$), and $|u \wedge v|$ is the area of the parallelogram spanned by u and v .

LEMMA 2.1 (Shoelace formula). *The area of a simple polygon $P_1P_2 \cdots P_m$ is $\frac{1}{2} \left| \sum_{j=1}^m P_j \wedge P_{j+1} \right|$, where $P_{m+1} = P_1$.*

We use this standard fact without proof; it follows from Green's theorem applied to the vector field $\frac{1}{2}(-y, x)$.

3 IDEA OF THE PROOF

Every statement is linear in the four points. Each side of $KLMN$ is the difference of two midpoints, and the terms that do not involve a diagonal cancel.

4 PROOF

Proof of (a).

$$L - K = \frac{B + C}{2} - \frac{A + B}{2} = \frac{C - A}{2}, \quad M - N = \frac{C + D}{2} - \frac{D + A}{2} = \frac{C - A}{2}.$$

The same computation with B and D in place of A and C gives $N - K = M - L = \frac{1}{2}(D - B)$. ■

Proof of (b). The midpoints of the diagonals KM and LN are

$$\frac{K + M}{2} = \frac{A + B + C + D}{4} = \frac{L + N}{2}. \quad \blacksquare$$

Proof of (c). By (a), the four sides have lengths $\frac{1}{2}AC$, $\frac{1}{2}BD$, $\frac{1}{2}AC$ and $\frac{1}{2}BD$. ■

LEMMA 4.1 (*Area from the diagonals*). For any $A, B, C, D \in \mathbb{R}^2$,

$$A \wedge B + B \wedge C + C \wedge D + D \wedge A = (C - A) \wedge (D - B).$$

PROOF. Expand the right-hand side by bilinearity and use antisymmetry:

$$(C - A) \wedge (D - B) = C \wedge D - C \wedge B - A \wedge D + A \wedge B = A \wedge B + B \wedge C + C \wedge D + D \wedge A. \quad \blacksquare$$

Proof of (d). By Lemmas 2.1 and 4.1, the area of $ABCD$ is $\frac{1}{2}|(C - A) \wedge (D - B)|$. By (a), $KLMN$ is the parallelogram spanned by $L - K = \frac{1}{2}(C - A)$ and $N - K = \frac{1}{2}(D - B)$, so its area is

$$\left| \frac{1}{2}(C - A) \wedge \frac{1}{2}(D - B) \right| = \frac{1}{4}|(C - A) \wedge (D - B)|,$$

which is half the area of $ABCD$. ■

5 REMARKS

Remark 5.1. The parallelogram degenerates to a segment exactly when the diagonals AC and BD are parallel, and to a point exactly when both diagonals collapse, that is, $A = C$ and $B = D$.

Remark 5.2. Corollaries follow from (a) at once. $KLMN$ is a rhombus exactly when $AC = BD$, and a rectangle exactly when $AC \perp BD$. So the midpoint quadrilateral of a rhombus is a rectangle, and that of a rectangle is a rhombus.

Remark 5.3. The coordinate proof in the Analytic Geometry notes places A at the origin and B on the x -axis. The vector proof needs no choice of coordinates, so it works in any dimension and for every quadrilateral at once.