

ANALYTIC GEOMETRY

Course notes

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CHAPTER 1: COORDINATE PROOFS

A coordinate proof of a statement in plane geometry has three steps. First, place the figure on the axes where the algebra is simplest, without losing generality. Then name the points with as few letters as possible. Finally, turn lengths, midpoints, parallelism and perpendicularity into algebra. Most of the care goes into the first step: a placement that quietly makes the figure special proves only a special case.

I DISTANCE AND DIVISION OF A SEGMENT

THEOREM 1.1 (*Distance formula*). The distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

THEOREM 1.2 (Division of a segment). Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be distinct points.

(a) The point P on P_1P_2 with $P_1P : PP_2 = r : s$ (where $r, s > 0$) is

$$P = \left(\frac{sx_1 + rx_2}{r+s}, \frac{sy_1 + ry_2}{r+s} \right).$$

In particular the midpoint is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2} \right)$.

(b) If $P(x, y)$ and $Q(x', y')$ also lie on the line P_1P_2 , then

$$\frac{PQ}{P_1P_2} = \frac{|x' - x|}{|x_2 - x_1|} \quad (\text{line not vertical}), \quad \frac{PQ}{P_1P_2} = \frac{|y' - y|}{|y_2 - y_1|} \quad (\text{line not horizontal}).$$

PROOF. Every point of the line is $P_1 + t(x_2 - x_1, y_2 - y_1)$ for exactly one real t , and by Theorem 1.1 the points with parameters t and t' are $|t' - t| \cdot P_1P_2$ apart. Their x -coordinates differ by $|t' - t||x_2 - x_1|$ and their y -coordinates by $|t' - t||y_2 - y_1|$, which gives (b). For (a), P lies between P_1 and P_2 at $t = r/(r+s)$, and $x_1 + \frac{r}{r+s}(x_2 - x_1) = \frac{sx_1 + rx_2}{r+s}$, and likewise for y . ■

Exercise 1.3 (Varignon). Prove that joining the midpoints of the sides of a quadrilateral, in order, gives a parallelogram.

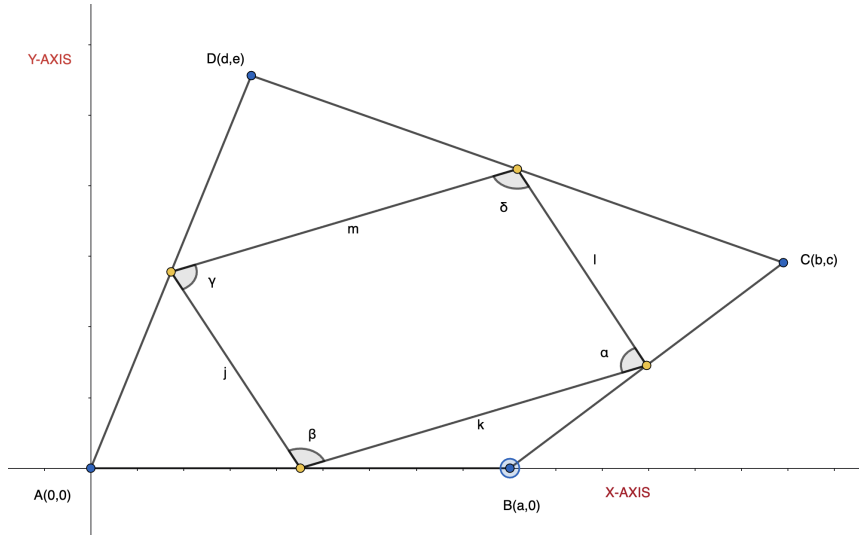


Figure 1: The quadrilateral $ABCD$ and its midpoint quadrilateral, with sides j, k, l, m .

PROOF. Place A at the origin and B on the positive x -axis, so that $A(0, 0)$, $B(a, 0)$, $C(b, c)$ and $D(d, e)$, as in Figure 1. Any quadrilateral can be moved into this position. By Theorem 1.2, the midpoints of AB , BC , CD and DA are

$$K\left(\frac{a}{2}, 0\right), \quad L\left(\frac{a+b}{2}, \frac{c}{2}\right), \quad M\left(\frac{b+d}{2}, \frac{c+e}{2}\right), \quad N\left(\frac{d}{2}, \frac{e}{2}\right).$$

Compare opposite sides of $KLMN$ by their changes in x and y :

$$\begin{aligned} K \rightarrow L : \left(\frac{b}{2}, \frac{c}{2} \right), & \quad N \rightarrow M : \left(\frac{b}{2}, \frac{c}{2} \right), \\ K \rightarrow N : \left(\frac{d-a}{2}, \frac{e}{2} \right), & \quad L \rightarrow M : \left(\frac{d-a}{2}, \frac{e}{2} \right). \end{aligned}$$

Each pair of opposite sides has the same change in x and the same change in y . By **Theorem 1.1**, opposite sides therefore have equal length: $k = m = \frac{1}{2}\sqrt{b^2 + c^2}$ and $j = l = \frac{1}{2}\sqrt{(d-a)^2 + e^2}$. They are also parallel, since they have equal slopes or are both vertical. A quadrilateral whose opposite sides are parallel is a parallelogram. ■

Remark 1.4. The proof shows more. KL and NM both have the same changes as $\frac{1}{2}\vec{AC} = (\frac{b}{2}, \frac{c}{2})$, so each is parallel to the diagonal AC and half its length. Likewise KN and LM are parallel to BD and half its length.

Exercise 1.5. Let M be the midpoint of the side AB of the parallelogram $ABCD$. Prove that the segment MC and the diagonal BD trisect each other.

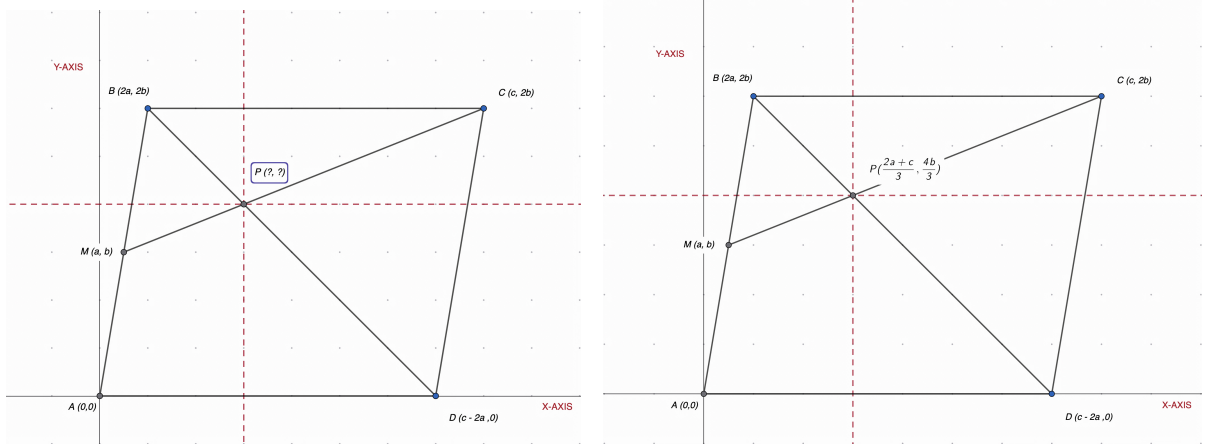


Figure 2: The parallelogram with A at the origin and D on the x -axis. Left: the unknown intersection P . Right: P found.

PROOF. Place the parallelogram as in **Figure 2**: $A(0, 0)$, $B(2a, 2b)$, $D(c-2a, 0)$ and $C(c, 2b)$, with $b > 0$ and $c \neq 2a$. Then $\vec{AB} = \vec{DC}$, so $ABCD$ really is a parallelogram. The factors of 2 make the midpoint $M(a, b)$ free of fractions.

Finding P . Suppose first that $c \neq a$ and $c \neq 4a$, so neither line is vertical. Then

$$MC : y = b + \frac{b}{c-a}(x-a), \quad BD : y = 2b + \frac{2b}{4a-c}(x-2a).$$

Set them equal, divide by b , and rearrange:

$$\frac{x-a}{c-a} - \frac{2(x-2a)}{4a-c} = 1.$$

Multiplying by $(c-a)(4a-c)$ gives

$$(x-a)(4a-c) - 2(x-2a)(c-a) = (c-a)(4a-c),$$

which expands to

$$3(2a-c)x - 8a^2 + 5ac = 5ac - c^2 - 4a^2.$$

So $3(2a-c)x = 4a^2 - c^2 = (2a-c)(2a+c)$. Since $c \neq 2a$,

$$x = \frac{2a+c}{3}.$$

Then $x-a = \frac{c-a}{3}$, and the equation of MC gives

$$y = b + \frac{b}{c-a} \cdot \frac{c-a}{3} = \frac{4b}{3}.$$

So the lines meet at $P(\frac{2a+c}{3}, \frac{4b}{3})$.

Ratios. Neither segment is horizontal, so by **Theorem 1.2(b)** lengths along each one are in the same ratio as the changes in y . Along BD , the y -coordinates of B, P, D are $2b, \frac{4b}{3}, 0$, so

$$BP : PD = \frac{2b}{3} : \frac{4b}{3} = 1 : 2.$$

Along MC , the y -coordinates of M, P, C are $b, \frac{4b}{3}, 2b$, so

$$MP : PC = \frac{b}{3} : \frac{2b}{3} = 1 : 2.$$

So P is a point of trisection of both segments: each segment cuts the other one-third of the way along. ■

Remark 1.6. **Theorem 1.2(a)** gives a shorter proof that also covers the cases $c = a$ and $c = 4a$ excluded above. The point one-third of the way from M to C is $\frac{2M+C}{3} = (\frac{2a+c}{3}, \frac{4b}{3})$. The point one-third of the way from B to D is $\frac{2B+D}{3} = (\frac{2a+c}{3}, \frac{4b}{3})$. They are the same point, so the segments meet there and trisect each other. No line equations are needed.

Exercise 1.7. Prove that the segment joining the midpoints of the non-parallel sides of a trapezoid is parallel to the parallel sides, and that its length is half the sum of their lengths.

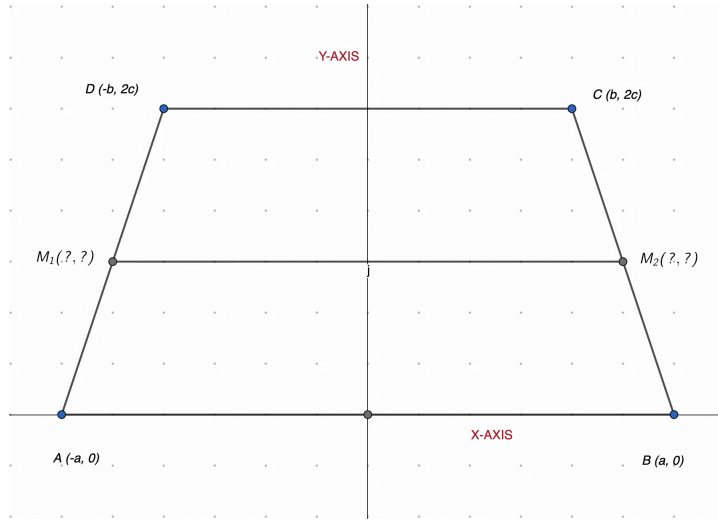


Figure 3: A trapezoid with $A(-a, 0), B(a, 0)$ and the top side at height $2c$. The figure shows the symmetric case $D(-b, 2c)$; the proof places D at $(d, 2c)$.

PROOF. Put the parallel sides on horizontal lines with AB centred at the origin: $A(-a, 0), B(a, 0), C(b, 2c)$ and $D(d, 2c)$, where $a, c > 0$ and $d < b$. The non-parallel sides are AD and BC , and by **Theorem 1.2** their midpoints are

$$M_1\left(\frac{d-a}{2}, c\right), \quad M_2\left(\frac{a+b}{2}, c\right).$$

Parallel. AB, DC and M_1M_2 all join two points with equal y -coordinates, so all three have slope 0 and are parallel.

Length. On horizontal segments the length is the change in x :

$$M_1M_2 = \frac{a+b}{2} - \frac{d-a}{2} = \frac{2a+(b-d)}{2}, \quad AB = 2a, \quad DC = b-d.$$

So $M_1M_2 = \frac{1}{2}(AB + DC)$. ■

Remark 1.8. The first version of this proof placed the top side symmetrically, at $D(-b, 2c)$ and $C(b, 2c)$ as in [Figure 3](#). That makes the figure symmetric about the y -axis, so it covers only isosceles trapezoids. Centring *one* parallel side costs nothing, but the other side must be left free. This is the placement trap described at the start of the chapter.

2 PERPENDICULAR LINES

THEOREM 2.1 (*Perpendicular lines*). Two non-vertical lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$. A vertical line is perpendicular exactly to the horizontal lines.

Exercise 2.2. Prove that the diagonals of a rhombus are perpendicular.

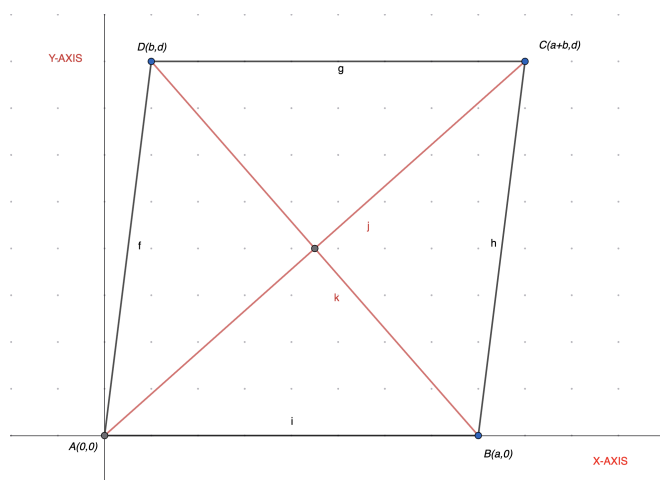


Figure 4: The rhombus $ABCD$ with its diagonals (red).

PROOF. Place the rhombus as in [Figure 4](#): $A(0, 0)$, $B(a, 0)$, $C(a + b, d)$ and $D(b, d)$, with $a > 0$ and $d \neq 0$. This is a parallelogram, and it is a rhombus when $AB = AD$, that is,

$$a^2 = b^2 + d^2.$$

This gives $b^2 - a^2 = -d^2 \neq 0$, so $b \neq \pm a$ and neither diagonal is vertical. Their slopes are

$$m_{AC} = \frac{d}{a+b}, \quad m_{BD} = \frac{d}{b-a},$$

and

$$m_{AC} m_{BD} = \frac{d^2}{b^2 - a^2} = \frac{d^2}{-d^2} = -1.$$

By [Theorem 2.1](#), the diagonals are perpendicular. ■

Exercise 2.3. Prove that the medians of an equilateral triangle are altitudes.

PROOF. Let $A(0, 0)$, $B(2a, 0)$ with $a > 0$, and $C(x, y)$ with $y > 0$, as in [Figure 5](#).

Locating C. The triangle is equilateral, so $CA^2 = AB^2 = 4a^2$ and $CB^2 = AB^2 = 4a^2$. By [Theorem 1.1](#),

$$x^2 + y^2 = 4a^2, \quad (x - 2a)^2 + y^2 = 4a^2.$$

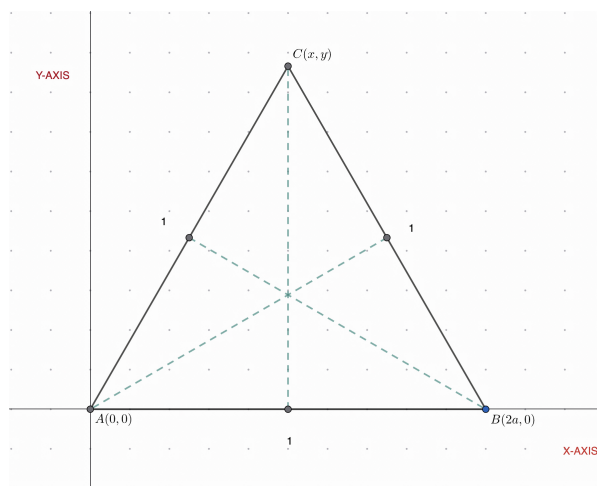


Figure 5: An equilateral triangle on the base AB , with its medians (dashed).

Subtracting the first equation from the second gives $-4ax + 4a^2 = 0$, so $x = a$. Then $y^2 = 3a^2$, and since $y > 0$, $C = (a, a\sqrt{3})$.

Midpoints. By Theorem 1.2,

$$M_{AB}(a, 0), \quad M_{AC}\left(\frac{a}{2}, \frac{a\sqrt{3}}{2}\right), \quad M_{BC}\left(\frac{3a}{2}, \frac{a\sqrt{3}}{2}\right).$$

Median from C. C and M_{AB} both have x -coordinate a , so this median is the vertical line $x = a$. It is perpendicular to the horizontal side AB .

Median from B. The slopes are

$$m_{AC} = \frac{a\sqrt{3}}{a} = \sqrt{3}, \quad m_{BM_{AC}} = \frac{a\sqrt{3}/2}{a/2 - 2a} = -\frac{1}{\sqrt{3}},$$

whose product is -1 .

Median from A. The slopes are

$$m_{BC} = \frac{a\sqrt{3}}{a - 2a} = -\sqrt{3}, \quad m_{AM_{BC}} = \frac{a\sqrt{3}/2}{3a/2} = \frac{1}{\sqrt{3}},$$

whose product is again -1 .

By Theorem 2.1, each median is perpendicular to the side it meets, so each median is an altitude. ■

Exercise 2.4 (Converse of Pythagoras). Prove that if the squares of two sides of a triangle add up to the square of the third side, then the triangle is a right triangle.

PROOF. Name the vertices so that the hypothesis reads $AB^2 + AC^2 = BC^2$, and put the shared vertex at the origin: $A(0, 0)$, $B(x_1, y_1)$ and $C(x_2, y_2)$. By Theorem 1.1,

$$(x_1^2 + y_1^2) + (x_2^2 + y_2^2) = (x_2 - x_1)^2 + (y_2 - y_1)^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2x_1x_2 - 2y_1y_2.$$

Cancelling the common terms leaves

$$x_1x_2 + y_1y_2 = 0. \quad (2.1)$$

We show that $AB \perp AC$.

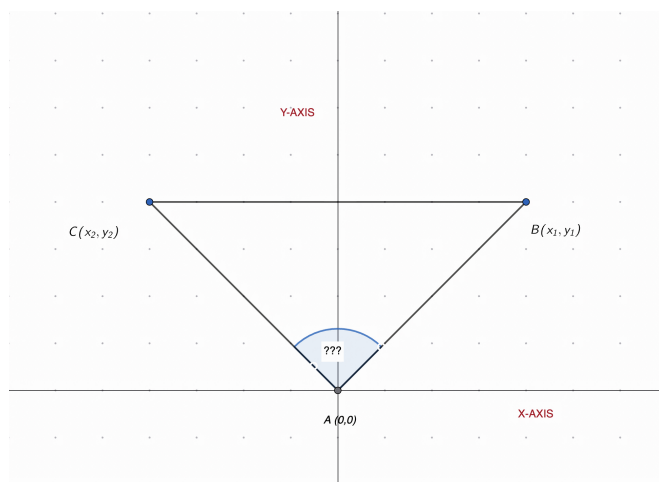


Figure 6: The triangle with the vertex in question at the origin, tilted off the axes so that nothing about the angle at A is assumed.

If $x_1x_2 \neq 0$, neither AB nor AC is vertical. Dividing (2.1) by x_1x_2 gives

$$\frac{y_1}{x_1} \cdot \frac{y_2}{x_2} = -1,$$

which is $m_{AB} m_{AC} = -1$. Apply **Theorem 2.1**.

If $x_1 = 0$, then $B \neq A$ forces $y_1 \neq 0$, and (2.1) becomes $y_1y_2 = 0$, so $y_2 = 0$. Then AB lies on the y -axis and AC on the x -axis, so they are perpendicular. The case $x_2 = 0$ is the same with B and C exchanged.

In every case the angle at A is a right angle. ■

Remark 2.5. The slope argument alone breaks down when a side is vertical, which is why the case split is needed. In vector language, (2.1) says $\vec{AB} \cdot \vec{AC} = 0$, which handles every case at once.

Exercise 2.6. Prove that the perpendicular bisector of a chord of a circle passes through the centre.

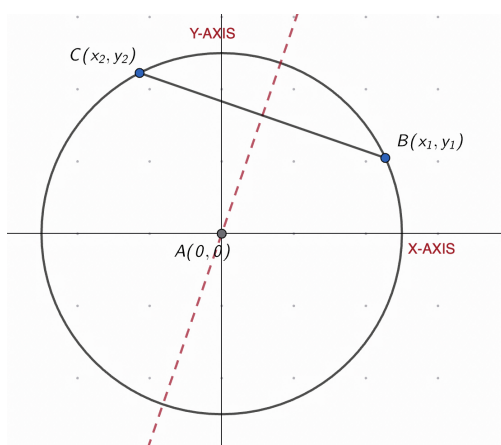


Figure 7: The chord BC and its perpendicular bisector (red), drawn without assuming that it passes through A .

PROOF. Centre the circle at the origin $A(0, 0)$ with radius r , and let the chord have endpoints $B(x_1, y_1) \neq C(x_2, y_2)$ on the circle, so

$$x_1^2 + y_1^2 = r^2 = x_2^2 + y_2^2. \quad (2.2)$$

The midpoint of BC is $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

General case: $x_1 \neq x_2$ and $y_1 \neq y_2$. Here $m_{BC} = \frac{y_2-y_1}{x_2-x_1}$ is neither zero nor undefined. By [Theorem 2.1](#), the perpendicular bisector has slope $-\frac{x_2-x_1}{y_2-y_1}$, so its equation is

$$y - \frac{y_1+y_2}{2} = -\frac{x_2-x_1}{y_2-y_1} \left(x - \frac{x_1+x_2}{2}\right).$$

The origin lies on this line exactly when

$$-\frac{y_1+y_2}{2} = \frac{(x_2-x_1)(x_2+x_1)}{2(y_2-y_1)}.$$

Multiplying by $-2(y_2-y_1) \neq 0$ turns this into the equivalent condition

$$y_2^2 - y_1^2 = -(x_2^2 - x_1^2), \quad \text{that is,} \quad x_2^2 + y_2^2 = x_1^2 + y_1^2.$$

This is true by (2.2). So the bisector passes through A .

Horizontal chord: $y_1 = y_2$. Then $x_1 \neq x_2$, and (2.2) gives $x_1^2 = x_2^2$, so $x_2 = -x_1$. The perpendicular bisector is the vertical line through M , namely $x = \frac{x_1+x_2}{2} = 0$. This is the y -axis, which contains A .

Vertical chord: $x_1 = x_2$. In the same way $y_2 = -y_1$, and the bisector is $y = 0$, which contains A . ■

Remark 2.7. A slope-free proof is possible. The perpendicular bisector of BC is exactly the set of points equidistant from B and C , and the centre is such a point, since $AB = AC = r$.

CHAPTER 2: CONICS AND THEIR REFLECTIVE PROPERTIES

Parabolas and ellipses in standard position. The tools are implicit differentiation for tangent slopes and the angle formula for comparing two lines. The chapter ends with the reflective property of the ellipse.

3 THE PARABOLA

THEOREM 3.1 (*Parabola in standard position*). The parabola with focus $(b+p, k)$ and directrix $x = b-p$, where $p \neq 0$, is

$$(y-k)^2 = 4p(x-b).$$

Its vertex is (b, k) and its axis is the line $y = k$. Exchanging the roles of x and y gives the parabola $(x-b)^2 = 4p(y-k)$ with a vertical axis.

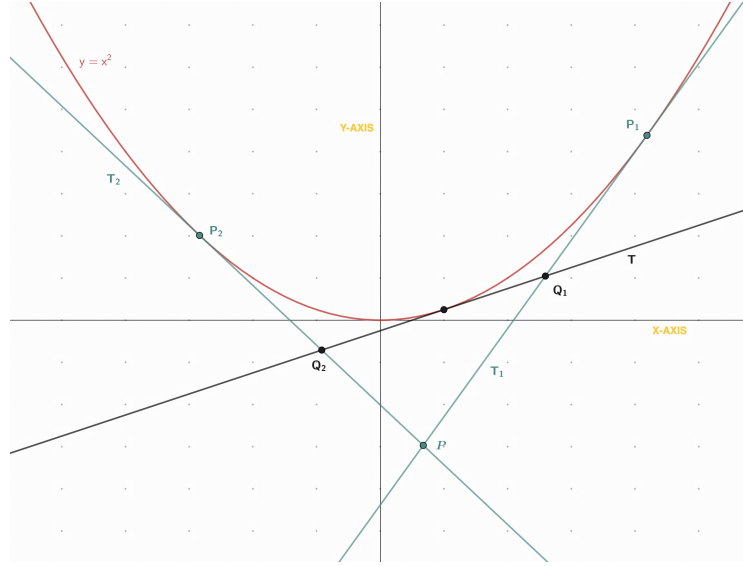
Exercise 3.2. Tangent lines T_1 and T_2 to the parabola $y = x^2$ at the points P_1 and P_2 meet at P . A third tangent T , touching the parabola at a point between P_1 and P_2 , meets T_1 at Q_1 and T_2 at Q_2 . Show that

$$\frac{PQ_1}{PP_1} + \frac{PQ_2}{PP_2} = 1.$$

Solution. Let $P_1 = (a, a^2)$ and $P_2 = (b, b^2)$, and let T touch at (c, c^2) with c strictly between a and b . In [Figure 8](#) we have $b < c < a$, but the argument below does not depend on the order.

Tangents. Since $\frac{dy}{dx} = 2x$, the tangent at (s, s^2) is $y - s^2 = 2s(x - s)$, that is, $y = 2sx - s^2$. So

$$T_1 : y = 2ax - a^2, \quad T_2 : y = 2bx - b^2, \quad T : y = 2cx - c^2.$$

Figure 8: Three tangents to $y = x^2$.

Intersections. Tangents at $s \neq u$ meet where $2(s-u)x = s^2 - u^2$, that is, at $x = \frac{s+u}{2}$, and then $y = 2s \cdot \frac{s+u}{2} - s^2 = su$. Hence

$$P\left(\frac{a+b}{2}, ab\right), \quad Q_1\left(\frac{a+c}{2}, ac\right), \quad Q_2\left(\frac{b+c}{2}, bc\right).$$

Ratios. The points P, Q_1, P_1 lie on T_1 , which is not vertical. By **Theorem 1.2(b)**, the ratio of lengths equals the ratio of changes in x :

$$\frac{PQ_1}{PP_1} = \frac{\left|\frac{a+b}{2} - \frac{a+c}{2}\right|}{\left|\frac{a+b}{2} - a\right|} = \frac{|b-c|}{|b-a|}.$$

In the same way, on T_2 ,

$$\frac{PQ_2}{PP_2} = \frac{\left|\frac{b+c}{2} - \frac{a+b}{2}\right|}{\left|b - \frac{a+b}{2}\right|} = \frac{|c-a|}{|b-a|}.$$

Because c lies between a and b , $|b-c| + |c-a| = |b-a|$, and so

$$\frac{PQ_1}{PP_1} + \frac{PQ_2}{PP_2} = \frac{|b-c| + |c-a|}{|b-a|} = 1. \quad \triangleleft$$

Remark 3.3. The intersection rule is worth remembering: the tangents at s and u meet at $\left(\frac{s+u}{2}, su\right)$, above the midpoint of s and u .

Exercise 3.4. Let $P(x_1, y_1)$ be a point other than the vertex on the parabola $y^2 = 2px$. Find the normal line at P , and show that it meets the x -axis at $Q(x_1 + p, 0)$.

Solution. Differentiating $y^2 = 2px$ implicitly gives $2y \frac{dy}{dx} = 2p$, so the tangent at P has slope p/y_1 . Here $y_1 \neq 0$ because P is not the vertex. By **Theorem 2.1**, the normal has slope $-y_1/p$, and its equation is

$$y - y_1 = -\frac{y_1}{p}(x - x_1).$$

At the x -axis $y = 0$, so $-y_1 = -\frac{y_1}{p}(x - x_1)$. Dividing by $-y_1 \neq 0$ gives $1 = \frac{x - x_1}{p}$, that is, $x = x_1 + p$. So the normal meets the axis at $Q(x_1 + p, 0)$. $\triangleleft$

Remark 3.5. This is the reflective property of the parabola in disguise. Writing $y^2 = 2px$ as $y^2 = 4 \cdot \frac{p}{2} \cdot x$, **Theorem 3.1** puts the focus at $F(\frac{p}{2}, 0)$ and the directrix at $x = -\frac{p}{2}$. By the focus-directrix definition, $FP =$

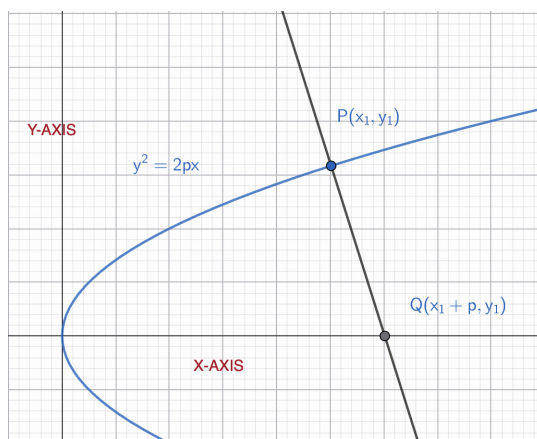


Figure 9: The normal at P . The label on Q should read $(x_1 + p, 0)$.

$x_1 + \frac{p}{2} = FQ$, so the triangle FPQ is isosceles. The normal PQ therefore makes equal angles with PF and with the horizontal line through P . In other words, rays parallel to the axis reflect through the focus.

4 THE ELLIPSE

THEOREM 4.1 (*Ellipse in standard position*). Let $0 \leq c < a$ and $b^2 = a^2 - c^2$. The set of points whose distances to the foci $F_1(-c, 0)$ and $F_2(c, 0)$ add up to $2a$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

It meets the x -axis at $(\pm a, 0)$ and the y -axis at $(0, \pm b)$.

Exercise 4.2. A highway overpass has the shape of half an ellipse, with a base 200 feet wide. The highway beneath it is 120 feet wide, and the minimum clearance over the highway is 40 feet. Find the height of the overpass at its centre.

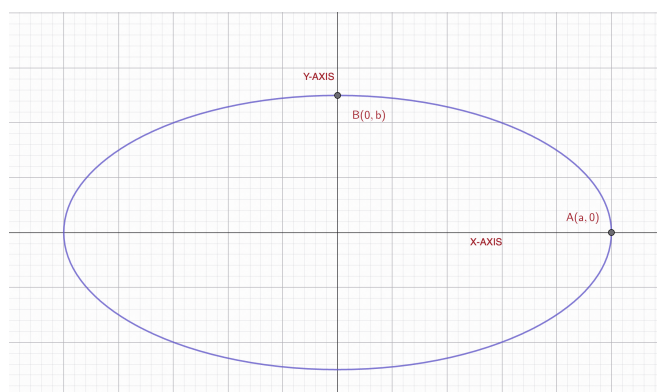


Figure 10: The overpass as the upper half of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. Put the centre of the base at the origin, so the arch is the upper half of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the height at the centre is b .

Base. The base is the major axis, so $2a = 200$ and $a = 100$.

Clearance. The highway runs from $x = -60$ to $x = 60$. On the upper half, y decreases as $|x|$ grows, so the

clearance over the road is smallest at its edges. Therefore $(60, 40)$ lies on the ellipse:

$$\frac{60^2}{100^2} + \frac{40^2}{b^2} = 1 \implies \frac{1600}{b^2} = 1 - 0.36 = 0.64 \implies b^2 = 2500.$$

Since $b > 0$, $b = 50$. The overpass is 50 feet high at its centre. ◀

5 REFLECTION

THEOREM 5.1 (Angle between two lines). Let l_1 and l_2 be non-perpendicular lines with slopes m_1 and m_2 , and let α be the angle from l_1 to l_2 (measured anticlockwise). Then

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

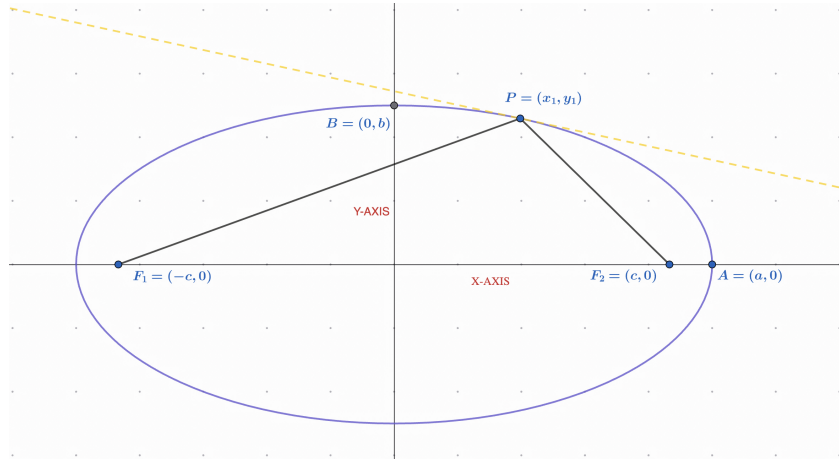


Figure 11: A ray from F_1 meets the ellipse at P and reflects to F_2 . The tangent at P is dashed.

THEOREM 5.2 (Reflective property of the ellipse). A ray leaving one focus of an ellipse is reflected by the ellipse through the other focus.

PROOF. Take the ellipse of Theorem 4.1 with $c > 0$ (if $c = 0$ it is a circle, and every ray from the centre returns along itself). Let $P(x_1, y_1)$ be on the ellipse.

If $y_1 = 0$, then $P = (\pm a, 0)$, the tangent there is vertical, and the ray runs along the x -axis, so it returns along the axis through both foci. Suppose now $y_1 \neq 0$ and, for the moment, $x_1 \neq \pm c$, so that neither focal line is vertical.

Slopes. Differentiating $x^2/a^2 + y^2/b^2 = 1$ implicitly gives $2x/a^2 + (2y/b^2)y' = 0$. So the tangent at P has slope

$$m = -\frac{b^2 x_1}{a^2 y_1},$$

and the lines F_1P and F_2P have slopes

$$m_1 = \frac{y_1}{x_1 + c}, \quad m_2 = \frac{y_1}{x_1 - c}.$$

Below we use $b^2 x_1^2 + a^2 y_1^2 = a^2 b^2$ (because P is on the ellipse) and $a^2 - b^2 = c^2$.

Incoming angle. Let α be the angle from F_1P to the tangent. By [Theorem 5.1](#), $\tan \alpha = (m - m_1)/(1 + mm_1)$. Multiply the numerator and the denominator by $a^2y_1(x_1 + c)$:

$$\begin{aligned}\text{numerator} &= -b^2x_1(x_1 + c) - a^2y_1^2 = -(b^2x_1^2 + a^2y_1^2) - b^2cx_1 = -b^2(a^2 + cx_1), \\ \text{denominator} &= a^2y_1(x_1 + c) - b^2x_1y_1 = y_1((a^2 - b^2)x_1 + a^2c) = cy_1(a^2 + cx_1).\end{aligned}$$

Since $|x_1| \leq a$ and $c < a$, we have $a^2 + cx_1 > 0$. So the denominator is not zero (the lines are not perpendicular), and

$$\tan \alpha = -\frac{b^2}{cy_1}.$$

Outgoing angle. Let β be the angle from the tangent to F_2P , so $\tan \beta = (m_2 - m)/(1 + mm_2)$. Multiply the numerator and the denominator by $a^2y_1(x_1 - c)$:

$$\begin{aligned}\text{numerator} &= a^2y_1^2 + b^2x_1(x_1 - c) = b^2(a^2 - cx_1), \\ \text{denominator} &= a^2y_1(x_1 - c) - b^2x_1y_1 = -cy_1(a^2 - cx_1),\end{aligned}$$

with $a^2 - cx_1 > 0$. Hence

$$\tan \beta = -\frac{b^2}{cy_1} = \tan \alpha.$$

Both angles lie in $[0, \pi)$, where $\tan$ is one-to-one, so $\alpha = \beta$. So F_1P and F_2P make equal angles with the tangent, one on each side: reflecting the line F_1P in the tangent gives the line F_2P . This is the law of reflection (the angle of incidence equals the angle of reflection), so the reflected ray passes through F_2 .

When $x_1 = \pm c$, one focal line is vertical and the slope formula does not apply. The result still holds there by continuity: both angles depend continuously on P , and they agree at every nearby point of the ellipse. ■

Remark 5.3. The simplifications in the proof rely on two facts: P lies on the ellipse, so $b^2x_1^2 + a^2y_1^2 = a^2b^2$, and $a^2 - b^2 = c^2$. Without substituting both, the expressions for $\tan \alpha$ and $\tan \beta$ do not visibly agree.

CHAPTER 3: ROTATION OF AXES

A second-degree equation with an xy term describes a conic whose axes are tilted. Rotating the coordinate axes through the right angle removes that term, and completing the square then puts the conic in standard position. The same formulas run backwards to take a conic from standard position to a tilted one.

6 REMOVING THE CROSS TERM

THEOREM 6.1 (Rotation of axes). Rotating the axes through an angle θ gives new coordinates (x', y') related to the old ones by

$$\begin{aligned}x &= x' \cos \theta - y' \sin \theta, & x' &= x \cos \theta + y \sin \theta, \\ y &= x' \sin \theta + y' \cos \theta, & y' &= -x \sin \theta + y \cos \theta.\end{aligned}$$

If $B \neq 0$, the rotation with

$$\cot 2\theta = \frac{A - C}{B}, \quad 0 < \theta < \frac{\pi}{2},$$

removes the cross term from $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. In particular $\theta = \pi/4$ when $A = C$.

PROOF. The two systems are inverse to each other, since the matrix of a rotation by θ has inverse the rotation by $-\theta$. Substitute the left-hand formulas into the quadratic part. Only x^2 , xy and y^2 produce an $x'y'$ term, and its coefficient is

$$-2A \sin \theta \cos \theta + B(\cos^2 \theta - \sin^2 \theta) + 2C \sin \theta \cos \theta = B \cos 2\theta - (A - C) \sin 2\theta.$$

This vanishes exactly when $\cot 2\theta = (A - C)/B$, and $\cot$ takes every real value once on $(0, \pi)$. ■

Exercise 6.2. Rotate the axes to remove the cross term from

$$x'^2 - 2x'y' + y'^2 - 2y' = 0,$$

and find a standard equation for the curve in the rotated $x''y''$ -coordinates.

Solution. Here $A = 1$, $B = -2$ and $C = 1$. Since $A = C$, **Theorem 6.1** gives $\theta = \frac{\pi}{4}$, with $\cos \theta = \sin \theta = \frac{1}{\sqrt{2}}$:

$$x' = \frac{x'' - y''}{\sqrt{2}}, \quad y' = \frac{x'' + y''}{\sqrt{2}}.$$

The first three terms form a perfect square, so the equation is $(x' - y')^2 - 2y' = 0$. Now

$$x' - y' = \frac{(x'' - y'') - (x'' + y'')}{\sqrt{2}} = -\sqrt{2}y'',$$

so the equation becomes

$$2y''^2 - \sqrt{2}(x'' + y'') = 0 \quad \text{or} \quad y''^2 - \frac{\sqrt{2}}{2}y'' = \frac{\sqrt{2}}{2}x''.$$

Complete the square by adding $(\frac{\sqrt{2}}{4})^2 = \frac{1}{8}$ to both sides:

$$\left(y'' - \frac{\sqrt{2}}{4}\right)^2 = \frac{\sqrt{2}}{2}x'' + \frac{1}{8} = \frac{\sqrt{2}}{2}\left(x'' + \frac{\sqrt{2}}{8}\right),$$

using $\frac{1}{8} \cdot \frac{2}{\sqrt{2}} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}$. By **Theorem 3.1**, this is a parabola with horizontal axis $y'' = \frac{\sqrt{2}}{4}$ and vertex $(-\frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{4})$. Since $4p = \frac{\sqrt{2}}{2}$, we have $p = \frac{\sqrt{2}}{8}$. ◁

7 ROTATING BACK

Exercise 7.1. An ellipse has foci $A(-12, -5)$ and $B(12, 5)$ and passes through $P_1(-2.5, 6)$. In coordinates whose x' -axis contains the foci, its equation is

$$\frac{4x'^2}{845} + \frac{4y'^2}{169} = 1.$$

Find its equation in the original xy -coordinates.

Solution. Angle. The x' -axis is the line through the origin and the foci. Its slope is $\frac{5-(-5)}{12-(-12)} = \frac{5}{12}$, so $\tan \theta = \frac{5}{12}$. With hypotenuse $\sqrt{5^2 + 12^2} = 13$, this gives $\sin \theta = \frac{5}{13}$ and $\cos \theta = \frac{12}{13}$.

Substitution. By **Theorem 6.1**,

$$x' = \frac{12x + 5y}{13}, \quad y' = \frac{-5x + 12y}{13}.$$

Multiplying the equation by 845 = 5 · 169 gives $4x'^2 + 20y'^2 = 845$. Substituting,

$$\frac{4(12x + 5y)^2 + 20(-5x + 12y)^2}{169} = 845.$$

Multiply by 169 (note $845 \cdot 169 = 142805$) and expand:

$$\begin{aligned} 4(144x^2 + 120xy + 25y^2) + 20(25x^2 - 120xy + 144y^2) &= 142805, \\ (576 + 500)x^2 + (480 - 2400)xy + (100 + 2880)y^2 &= 142805. \end{aligned}$$

The ellipse is

$$1076x^2 - 1920xy + 2980y^2 = 142805.$$

◁

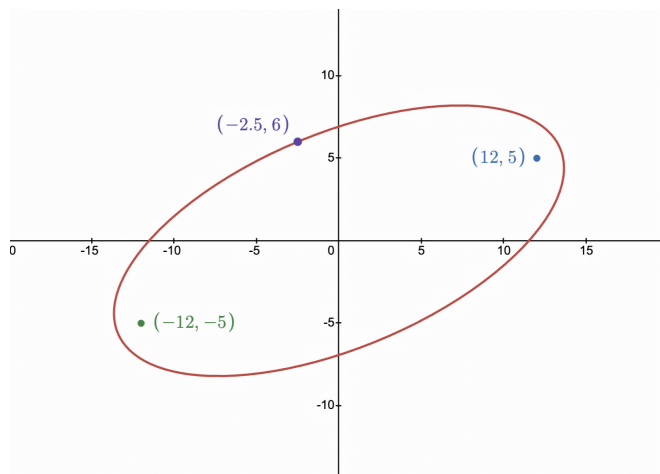


Figure 12: The ellipse $1076x^2 - 1920xy + 2980y^2 = 142805$, with its foci and the point P_1 .

Remark 7.2. Three checks, all by hand.

- *The given point.* At $P_1(-2.5, 6)$, $x' = \frac{-30+30}{13} = 0$ and $y' = \frac{12.5+72}{13} = 6.5$, so $4x'^2 + 20y'^2 = 20 \cdot 42.25 = 845$. In the final equation, $1076(6.25) + 1920(15) + 2980(36) = 6725 + 28800 + 107280 = 142805$.
- *The foci.* In the $x'y'$ form, $a^2 = \frac{845}{4}$ and $b^2 = \frac{169}{4}$, so $c^2 = a^2 - b^2 = 169$ and $c = 13$. This is the distance from the origin to $(12, 5)$, as it should be.
- *The picture.* Figure 12 plots the result.

CHAPTER 4: PARAMETRIC CURVES

A parametric curve gives x and y separately as functions of a parameter t . This records not just the set of points but the way the curve is traced: where it starts, which way it goes and how often it passes each point. Circles are the basic building block. The chapter ends with the geometry behind the rotary engine.

8 CIRCLES AND ARCS

THEOREM 8.1 (*Parametric circle*). For $0 \leq t \leq 1$,

$$x(t) = h + r \cos 2\pi t, \quad y(t) = k + r \sin 2\pi t$$

traces the circle with centre (h, k) and radius r exactly once, anticlockwise, starting and ending at $(h + r, k)$. Replacing $2\pi t$ by $\theta_0 + (\theta_1 - \theta_0)t$ traces only the arc from angle θ_0 to angle θ_1 .

Exercise 8.2. Find parametric equations, with $0 \leq t \leq 1$, for a particle that moves along $y = x^2$ from $(1, 1)$ to $(-1, 1)$ and back to $(1, 1)$.

Solution. Take

$$x(t) = \cos 2\pi t, \quad y(t) = \cos^2 2\pi t, \quad 0 \leq t \leq 1.$$

Then $y = x^2$ for every t , so the particle stays on the parabola. As t runs over $[0, \frac{1}{2}]$, $2\pi t$ runs over $[0, \pi]$ and $x = \cos 2\pi t$ decreases from 1 to -1 . As t runs over $[\frac{1}{2}, 1]$, x increases back to 1. So the particle goes from $(1, 1)$ to $(-1, 1)$ and returns, passing through each point of the arc once in each direction. $\triangleleft$

Exercise 8.3. Find parametric equations, with $0 \leq t \leq 1$ (several sets are needed), that draw the face in Figure 13.

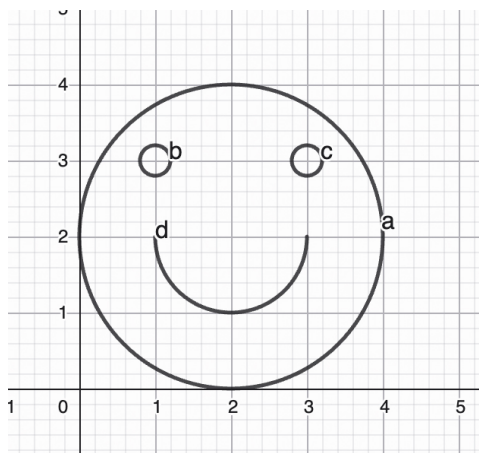


Figure 13: The face, drawn from the four sets of equations below.

Solution. All four pieces come from Theorem 8.1.

- (a) Face, centre $(2, 2)$ and radius 2: $x = 2 + 2 \cos 2\pi t, y = 2 + 2 \sin 2\pi t$.
- (b) Left eye, centre $(1, 3)$ and radius 0.2: $x = 1 + 0.2 \cos 2\pi t, y = 3 + 0.2 \sin 2\pi t$.
- (c) Right eye, centre $(3, 3)$ and radius 0.2: $x = 3 + 0.2 \cos 2\pi t, y = 3 + 0.2 \sin 2\pi t$.
- (d) Smile, the lower half of the circle with centre $(2, 2)$ and radius 1, so the angles run from π to 2π :

$$x = 2 + \cos(\pi + \pi t), \quad y = 2 + \sin(\pi + \pi t).$$

It starts at $(1, 2)$ when $t = 0$, passes through $(2, 1)$ when $t = \frac{1}{2}$ and ends at $(3, 2)$ when $t = 1$. $\triangleleft$

Exercise 8.4. Circles of radius a and b , with $a > b > 0$, are centred at the origin. A ray from the origin at angle t to the positive x -axis meets the large circle and then the small one. The point P takes its x -coordinate from the first intersection and its y -coordinate from the second. Find parametric equations for the curve traced by P , and identify it.

Solution. The ray at angle t meets the circle of radius a at $(a \cos t, a \sin t)$, and it meets the circle of radius b at $(b \cos t, b \sin t)$. Hence

$$x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi.$$

Since $\cos^2 t + \sin^2 t = 1$, eliminating t gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the ellipse of Theorem 4.1 with semi-axes a and b . As t runs over $[0, 2\pi]$, P goes once around it, anticlockwise. $\triangleleft$

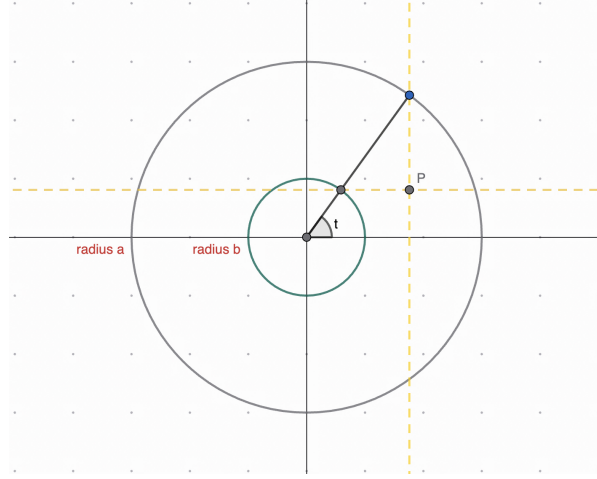


Figure 14: P takes its x -coordinate from the large circle and its y -coordinate from the small one.

9 THE ROTARY ENGINE

THEOREM 9.1 (Centroid). *The three medians of the triangle with vertices (x_i, y_i) , $i = 1, 2, 3$, meet at the centroid*

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right),$$

which lies two-thirds of the way along each median from its vertex.

Exercise 9.2. Show that an equilateral triangle can be inscribed in the epitrochoid

$$x = 3r \cos t + b \cos 3t, \quad y = 3r \sin t + b \sin 3t,$$

and that as the triangle moves its centroid traces a circle of radius b about the centre of the epitrochoid.

PROOF. This is the principle of the rotary (Wankel) engine: a triangular rotor turns with its three tips on an epitrochoid-shaped housing, while its centre moves on a small circle.

Vertices. Fix t and take the three points of the curve at the parameters $t_k = t + \frac{2\pi k}{3}$, for $k = 0, 1, 2$. Since $3t_k = 3t + 2\pi k$, the b -terms are the same for all three points: $b \cos 3t_k = b \cos 3t$ and $b \sin 3t_k = b \sin 3t$. Writing $G = (b \cos 3t, b \sin 3t)$, the vertices are

$$V_k = G + 3r (\cos t_k, \sin t_k), \quad k = 0, 1, 2.$$

Equilateral. The points $3r(\cos t_k, \sin t_k)$ lie on the circle of radius $3r$ about the origin, spaced $\frac{2\pi}{3}$ apart, so they form an equilateral triangle with side $2 \cdot 3r \sin \frac{\pi}{3} = 3\sqrt{3}r$. Adding the fixed point G to all three is a translation, which preserves distances. So $V_0V_1V_2$ is equilateral, and its vertices lie on the epitrochoid.

Centroid. By Theorem 9.1, the centroid is $G + r \sum_k (\cos t_k, \sin t_k)$. By the addition formulas, with $\cos \frac{2\pi}{3} = \cos \frac{4\pi}{3} = -\frac{1}{2}$ and $\sin \frac{2\pi}{3} = \sin \frac{4\pi}{3} = \frac{\sqrt{3}}{2}$,

$$\sum_{k=0}^2 \cos t_k = \cos t + \left(-\frac{1}{2} \cos t - \frac{\sqrt{3}}{2} \sin t \right) + \left(-\frac{1}{2} \cos t + \frac{\sqrt{3}}{2} \sin t \right) = 0,$$

and likewise $\sum_k \sin t_k = 0$. So the centroid is

$$G = (b \cos 3t, b \sin 3t),$$

which, as t varies, runs around the circle of radius b centred at the origin. ■

Remark 9.3. The vertices turn at rate 1 in t , while the centroid goes round at rate 3. So the rotor's centre makes three orbits for every turn of the rotor, which is the 3:1 gearing of the Wankel engine.