

REFLECTIVE PROPERTY OF THE ELLIPSE

Analytic Geometry, Theorem 5.2

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THEOREM 5.2 (*Reflective property of the ellipse*). *A ray leaving one focus of an ellipse is reflected by the ellipse through the other focus.*

PROOF. Take the ellipse of [Theorem 4.1](#) with $c > 0$ (if $c = 0$ it is a circle, and every ray from the centre returns along itself). Let $P(x_1, y_1)$ be on the ellipse.

If $y_1 = 0$, then $P = (\pm a, 0)$, the tangent there is vertical, and the ray runs along the x -axis, so it returns along the axis through both foci. Suppose now $y_1 \neq 0$ and, for the moment, $x_1 \neq \pm c$, so that neither focal line is vertical.

Slopes. Differentiating $x^2/a^2 + y^2/b^2 = 1$ implicitly gives $2x/a^2 + (2y/b^2)y' = 0$. So the tangent at P has slope

$$m = -\frac{b^2 x_1}{a^2 y_1},$$

and the lines F_1P and F_2P have slopes

$$m_1 = \frac{y_1}{x_1 + c}, \quad m_2 = \frac{y_1}{x_1 - c}.$$

Below we use $b^2 x_1^2 + a^2 y_1^2 = a^2 b^2$ (because P is on the ellipse) and $a^2 - b^2 = c^2$.

Incoming angle. Let α be the angle from F_1P to the tangent. By [Theorem 5.1](#), $\tan \alpha = (m - m_1)/(1 + mm_1)$. Multiply the numerator and the denominator by $a^2 y_1(x_1 + c)$:

$$\begin{aligned} \text{numerator} &= -b^2 x_1(x_1 + c) - a^2 y_1^2 = -(b^2 x_1^2 + a^2 y_1^2) - b^2 c x_1 = -b^2(a^2 + c x_1), \\ \text{denominator} &= a^2 y_1(x_1 + c) - b^2 x_1 y_1 = y_1((a^2 - b^2)x_1 + a^2 c) = c y_1(a^2 + c x_1). \end{aligned}$$

Since $|x_1| \leq a$ and $c < a$, we have $a^2 + c x_1 > 0$. So the denominator is not zero (the lines are not perpendicular), and

$$\tan \alpha = -\frac{b^2}{c y_1}.$$

Outgoing angle. Let β be the angle from the tangent to F_2P , so $\tan \beta = (m_2 - m)/(1 + mm_2)$. Multiply the numerator and the denominator by $a^2 y_1(x_1 - c)$:

$$\begin{aligned} \text{numerator} &= a^2 y_1^2 + b^2 x_1(x_1 - c) = b^2(a^2 - c x_1), \\ \text{denominator} &= a^2 y_1(x_1 - c) - b^2 x_1 y_1 = -c y_1(a^2 - c x_1), \end{aligned}$$

with $a^2 - c x_1 > 0$. Hence

$$\tan \beta = -\frac{b^2}{c y_1} = \tan \alpha.$$

Both angles lie in $[0, \pi)$, where $\tan$ is one-to-one, so $\alpha = \beta$. So F_1P and F_2P make equal angles with the tangent, one on each side: reflecting the line F_1P in the tangent gives the line F_2P . This is the law of reflection (the angle of incidence equals the angle of reflection), so the reflected ray passes through F_2 .

When $x_1 = \pm c$, one focal line is vertical and the slope formula does not apply. The result still holds there by continuity: both angles depend continuously on P , and they agree at every nearby point of the ellipse. ■