

CLASSICAL MECHANICS

Reading notes on Arnold

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PREFACE

Several mathematicians led me to this subject. The first is Vladimir Arnold, whose work I met while studying abstract algebra in my junior year of college. The second is Eric Weinstein, and the third is Roger Penrose.

My interest began when I became aware of the stagnation in fundamental physics. Vast sums have gone into the search for a quantum theory of gravity, and, as I see it, we are hardly closer to one than we were sixty years ago. To find out where things went south, I first had to understand which way is north. That starts with a bridge between mathematics and classical mechanics, and these notes are my attempt to cross it. I hope they help you as they have helped me.

The notes follow Arnold's *Mathematical Methods of Classical Mechanics* [1]. The framework and the principles are his; the expansions and worked details are mine.

CHAPTER 1: FIRST PRINCIPLES

Mechanics rests on two facts taken from experiment: Galileo's principle of relativity and Newton's principle of determinacy [1, ch. 1]. Before stating them we need the stage on which they act.

I SPACE AND TIME

Space is three-dimensional and Euclidean. Time is one-dimensional. The second word needs a definition.

DEFINITION 1.1 (*Euclidean space*). The scalar product on $\mathbb{R}^n$ is $(\mathbf{x}, \mathbf{y}) = x_1 y_1 + \cdots + x_n y_n$. It gives the distance

$$\rho(\mathbf{x}, \mathbf{y}) = \sqrt{(\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y})}.$$

$\mathbb{R}^n$ with this distance is *Euclidean space* E^n . A space is *Euclidean* if its distances behave like these, so that lengths and angles are measured as in school geometry.

2 THE PRINCIPLE OF RELATIVITY

DEFINITION 2.1 (*Galileo's principle of relativity*). There exist coordinate systems, called *inertial*, with two properties.

1. All the laws of nature are the same, at every moment of time, in every inertial coordinate system.
2. Every coordinate system in uniform rectilinear motion relative to an inertial one is itself inertial.

Example 2.2. Suppose a coordinate system attached to the earth is inertial. Then an experimenter in a train moving uniformly in a straight line relative to the earth cannot detect the train's motion by any experiment carried out inside the carriage.

Remark 2.3. In fact a coordinate system attached to the earth is only approximately inertial: the earth rotates and moves on a curved orbit. The principle is an idealisation. Our equations describe real motions only approximately, so bodies moving on the earth cannot strictly be called inertial.

3 THE PRINCIPLE OF DETERMINACY

DEFINITION 3.1 (*Newton's principle of determinacy*). The initial state of a mechanical system, meaning all positions and all velocities of its points at one moment, uniquely determines all of its motion.

The principle says that accelerations are functions of positions, velocities and time. Writing this down gives the equation of motion.

THEOREM 3.2 (*Newton's equation*). Let a mechanical system of n points have configuration $\mathbf{x}(t) \in \mathbb{R}^N$, where $N = 3n$. By the determinacy principle (*Definition 3.1*) there is a function $\mathbf{F} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N$ such that every motion of the system satisfies

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t).$$

Conversely, if $\mathbf{F}$ is smooth, then a solution is uniquely determined by $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$.

Source: [I, ch. I]

Remark 3.3. Experience shows that the state at a moment also needs *velocities*: a stone dropped and a stone thrown from the same point move differently. Experience also shows that velocities are *enough*: we do not need to know accelerations in advance. That is why the equation has second order.

CHAPTER 2: THE GALILEAN GROUP

To turn the principle of relativity into mathematics, we need a precise model of space-time and of the transformations that preserve its structure [I, ch. I].

4 AFFINE SPACE

DEFINITION 4.1 (Affine space). Affine n -space A^n is a set of points on which the vector space $\mathbb{R}^n$ acts by parallel displacement. To a point $a \in A^n$ and a vector $\xi \in \mathbb{R}^n$ corresponds the point $a + \xi$, and for any two points a, b there is exactly one vector $b - a \in \mathbb{R}^n$ carrying a to b .

Remark 4.2. A^n differs from $\mathbb{R}^n$ in having no distinguished origin. Points can be subtracted, giving a vector, but not added. Choosing an origin $o \in A^n$ identifies A^n with $\mathbb{R}^n$ through $a \mapsto a - o$, but no choice is preferred. This is exactly the situation in physical space.

5 GALILEAN STRUCTURE

DEFINITION 5.1 (Galilean space-time). A Galilean structure consists of three things.

1. The *universe*, a four-dimensional affine space A^4 , whose points are called *events*.
2. *Time*, a linear map $t : \mathbb{R}^4 \rightarrow \mathbb{R}$ from the displacements of the universe onto the real line. The *time interval* from event a to event b is $t(b - a)$. Events with $t(b - a) = 0$ are *simultaneous*.
3. A *distance* between simultaneous events, $\rho(a, b) = \sqrt{(a - b, a - b)}$, given by a scalar product on the three-dimensional kernel of t . With it, each set of simultaneous events is a Euclidean space E^3 .

The coordinate model is $\mathbb{R} \times \mathbb{R}^3$, with time t and the standard distance on each slice $\{t\} \times \mathbb{R}^3$.

DEFINITION 5.2 (Galilean group). A Galilean transformation is an affine map of A^4 to itself that preserves time intervals and distances between simultaneous events. These transformations form the *Galilean group*.

Exercise 5.3. Check that the following maps of $\mathbb{R} \times \mathbb{R}^3$ are Galilean:

- (a) a *uniform motion* with velocity $\mathbf{v}$, $g_1(t, \mathbf{x}) = (t, \mathbf{x} + \mathbf{v}t)$;
- (b) a *translation*, $g_2(t, \mathbf{x}) = (t + s, \mathbf{x} + \mathbf{s})$;
- (c) a *rotation*, $g_3(t, \mathbf{x}) = (t, G\mathbf{x})$, where $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is orthogonal.

Solution. All three are affine. Take events $(t, \mathbf{x})$ and $(t', \mathbf{x}')$.

- (a) The time interval is unchanged. If $t = t'$, the spatial difference $(\mathbf{x}' + \mathbf{v}t) - (\mathbf{x} + \mathbf{v}t) = \mathbf{x}' - \mathbf{x}$ is unchanged too.
- (b) Both times and both positions are shifted by the same amounts, so differences are unchanged.
- (c) Times are unchanged, and $|G\mathbf{x}' - G\mathbf{x}| = |G(\mathbf{x}' - \mathbf{x})| = |\mathbf{x}' - \mathbf{x}|$ because G is orthogonal. ◀

THEOREM 5.4 (Structure of the Galilean group). Every Galilean transformation g of $\mathbb{R} \times \mathbb{R}^3$ can be written uniquely as

$$g = g_1 \circ g_2 \circ g_3,$$

where g_1 is a uniform motion, g_2 a translation and g_3 a rotation (*Exercise 5.3*).

Source: [1, ch. 1]

PROOF. A Galilean transformation is affine, so $g(t, \mathbf{x}) = L(t, \mathbf{x}) + (s, \mathbf{s})$ for a linear map L and a fixed $(s, \mathbf{s}) \in \mathbb{R} \times \mathbb{R}^3$.

Time. g preserves time intervals, so the time component of $L(\tau, \xi)$ is τ .

Space. Hence L sends a simultaneous displacement $(0, \xi)$ to a simultaneous displacement $(0, G\xi)$ for some linear $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Distances between simultaneous events are preserved, so $|G\xi| = |\xi|$, and G is orthogonal.

Velocity. Write $L(1, \mathbf{0}) = (1, \mathbf{v})$. By linearity $L(\tau, \xi) = (\tau, \mathbf{v}\tau + G\xi)$, so

$$g(t, \mathbf{x}) = (t + s, G\mathbf{x} + \mathbf{v}t + \mathbf{s}).$$

Factorisation. For $g_1(t, \mathbf{x}) = (t, \mathbf{x} + \mathbf{v}t)$, $g_2(t, \mathbf{x}) = (t + s, \mathbf{x} + \mathbf{u})$ and $g_3(t, \mathbf{x}) = (t, G\mathbf{x})$,

$$g_1 g_2 g_3(t, \mathbf{x}) = (t + s, G\mathbf{x} + \mathbf{u} + \mathbf{v}(t + s)).$$

This equals g exactly when $\mathbf{u} = \mathbf{s} - \mathbf{v}s$.

Uniqueness. The composite determines s (its time shift), G (its action on $\mathbf{x}$), $\mathbf{v}$ (its coefficient of t) and then $\mathbf{u}$. So two factorisations of the same g coincide. ■

Remark 5.5. Counting parameters: $\mathbf{v}$ gives 3, $(s, \mathbf{s})$ gives 4, and the orthogonal G gives 3 (three angles, plus a choice of sign for $\det G$). So the Galilean group is 10-dimensional.

6 RELATIVITY IN THE EQUATIONS

In this language, Galileo's principle says that Newton's equation ([Theorem 3.2](#)) is unchanged by Galilean transformations: if g is Galilean and $\mathbf{x}(t)$ is a motion, so is its image under g . Each part of the group then forces a property of $\mathbf{F}$.

Exercise 6.1. Use invariance under the time translations $(t, \mathbf{x}) \mapsto (t + s, \mathbf{x})$ to show that $\mathbf{F}$ does not depend on t .

PROOF. Let $\mathbf{x}(t)$ be a motion. Its image under the translation is the motion $\mathbf{y}(t) = \mathbf{x}(t - s)$, which by assumption satisfies the same equation:

$$\ddot{\mathbf{y}}(t) = \mathbf{F}(\mathbf{y}(t), \dot{\mathbf{y}}(t), t).$$

But $\ddot{\mathbf{y}}(t) = \ddot{\mathbf{x}}(t - s) = \mathbf{F}(\mathbf{x}(t - s), \dot{\mathbf{x}}(t - s), t - s)$. Comparing the two,

$$\mathbf{F}(\mathbf{x}_0, \mathbf{v}_0, t) = \mathbf{F}(\mathbf{x}_0, \mathbf{v}_0, t - s),$$

where $\mathbf{x}_0 = \mathbf{x}(t - s)$ and $\mathbf{v}_0 = \dot{\mathbf{x}}(t - s)$. By determinacy there is a motion through any initial state $(\mathbf{x}_0, \mathbf{v}_0)$ at any time. So the identity holds for all arguments and all s , and $\mathbf{F} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$. ■

Remark 6.2. In the same way, invariance under spatial translations forces $\mathbf{F}$ to depend only on the relative positions $\mathbf{x}_i - \mathbf{x}_j$. Uniform motions force it to depend only on the relative velocities. Rotations make it equivariant: rotating all positions and velocities rotates the accelerations.

REFERENCES

- [1] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed., trans. K. Vogtmann and A. Weinstein, Graduate Texts in Mathematics 60, Springer, New York, 1989.