

STRUCTURE OF THE GALILEAN GROUP

Classical Mechanics, Theorem 5.4

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THEOREM 5.4 (*Structure of the Galilean group*). Every Galilean transformation g of $\mathbb{R} \times \mathbb{R}^3$ can be written uniquely as

$$g = g_1 \circ g_2 \circ g_3,$$

where g_1 is a uniform motion, g_2 a translation and g_3 a rotation ([Exercise 5.3](#)).

Source: [1, ch. 1]

PROOF. A Galilean transformation is affine, so $g(t, \mathbf{x}) = L(t, \mathbf{x}) + (s, \mathbf{s})$ for a linear map L and a fixed $(s, \mathbf{s}) \in \mathbb{R} \times \mathbb{R}^3$.

Time. g preserves time intervals, so the time component of $L(\tau, \xi)$ is τ .

Space. Hence L sends a simultaneous displacement $(0, \xi)$ to a simultaneous displacement $(0, G\xi)$ for some linear $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Distances between simultaneous events are preserved, so $|G\xi| = |\xi|$, and G is orthogonal.

Velocity. Write $L(1, \mathbf{0}) = (1, \mathbf{v})$. By linearity $L(\tau, \xi) = (\tau, \mathbf{v}\tau + G\xi)$, so

$$g(t, \mathbf{x}) = (t + s, G\mathbf{x} + \mathbf{v}t + \mathbf{s}).$$

Factorisation. For $g_1(t, \mathbf{x}) = (t, \mathbf{x} + \mathbf{v}t)$, $g_2(t, \mathbf{x}) = (t + s, \mathbf{x} + \mathbf{u})$ and $g_3(t, \mathbf{x}) = (t, G\mathbf{x})$,

$$g_1 g_2 g_3(t, \mathbf{x}) = (t + s, G\mathbf{x} + \mathbf{u} + \mathbf{v}(t + s)).$$

This equals g exactly when $\mathbf{u} = \mathbf{s} - \mathbf{v}s$.

Uniqueness. The composite determines s (its time shift), G (its action on $\mathbf{x}$), $\mathbf{v}$ (its coefficient of t) and then $\mathbf{u}$. So two factorisations of the same g coincide. ■

REFERENCES

- [1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed., trans. K. Vogtmann and A. Weinstein, Graduate Texts in Mathematics 60, Springer, New York, 1989.