

COMPLEX ANALYSIS

Master's notes

Drew M. Schaffner

CONTENTS

CHAPTER 1: CAUCHY'S INTEGRAL FORMULA AND LIOUVILLE'S THEOREM	I
1 Cauchy's theorem	2
2 Cauchy's integral formula	2
3 Cauchy's estimates and Liouville's theorem	4
4 The maximum modulus principle	5
CHAPTER 2: POWER AND TAYLOR SERIES	5
5 Radius of convergence	6
6 Termwise differentiation	6
7 Uniqueness of coefficients	8
8 Products and Taylor's theorem	9
CHAPTER 3: SINGULARITIES, LAURENT SERIES AND RESIDUES	10
9 Laurent series	10
10 Removable singularities	11
11 Zeros and poles	11
12 Residues	12
CHAPTER 4: REAL INTEGRALS BY RESIDUES	13
13 Integrals over a period	14
CHAPTER 5: REAL AND COMPLEX DIFFERENTIABILITY	15
14 One derivative gives all of them	15
15 No mean value theorem	15

CHAPTER 1: CAUCHY'S INTEGRAL FORMULA AND LIOUVILLE'S THEOREM

An analytic function is pinned down by its values on a closed curve: Cauchy's integral formula recovers f and all its derivatives inside from an integral around the edge. Bounding that integral gives Cauchy's estimates, and from them Liouville's theorem and the polynomial growth results.

I CAUCHY'S THEOREM

THEOREM 1.1 (ML-inequality). *If f is continuous on a contour Γ of length L and $|f(z)| \leq M$ for all z on Γ , then*

$$\left| \int_{\Gamma} f(z) \, dz \right| \leq ML.$$

Source: [1, ch. 4]

THEOREM 1.2 (Cauchy's integral theorem). *If f is analytic in a simply connected domain D and Γ is any closed contour in D , then*

$$\oint_{\Gamma} f(z) \, dz = 0.$$

Source: [1, ch. 4]

THEOREM 1.3 (Deformation of contours). *Let C be a simple closed positively oriented contour, and let $C_1, \dots, C_n$ be simple closed positively oriented contours inside C whose interiors are pairwise disjoint. If f is analytic on a domain containing all these contours and the region inside C but outside every C_k , then*

$$\oint_C f(z) \, dz = \sum_{k=1}^n \oint_{C_k} f(z) \, dz.$$

Source: [1, ch. 4]

2 CAUCHY'S INTEGRAL FORMULA

THEOREM 2.1 (Cauchy's integral formula). *Let f be analytic in a simply connected domain D , and let Γ be a simple closed positively oriented contour in D . For every z_0 inside Γ ,*

$$f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dz.$$

Source: [1, ch. 4]

PROOF. Let C_r be the circle $|z - z_0| = r$, with r small enough that C_r lies inside Γ . The integrand is analytic everywhere except at z_0 , so by **Theorem 1.3**

$$\oint_{\Gamma} \frac{f(z)}{z - z_0} \, dz = \oint_{C_r} \frac{f(z)}{z - z_0} \, dz = f(z_0) \oint_{C_r} \frac{dz}{z - z_0} + \oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz.$$

Parametrising $z = z_0 + re^{it}$, $0 \leq t \leq 2\pi$, the first integral is $\int_0^{2\pi} i \, dt = 2\pi i$. Let E_r be the second. On C_r its integrand is at most $\max_{C_r} |f(z) - f(z_0)|/r$ in modulus, so by **Theorem 1.1**

$$|E_r| \leq 2\pi \max_{|z - z_0| = r} |f(z) - f(z_0)| \longrightarrow 0 \quad \text{as } r \rightarrow 0,$$

by continuity of f at z_0 . But $E_r = \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dz - 2\pi i f(z_0)$ does not depend on r , so $E_r = 0$. ■

THEOREM 2.2 (*Cauchy's integral formula for derivatives*). Under the hypotheses of Theorem 2.1, f has derivatives of every order in D , each of them analytic in D , and for every z_0 inside Γ

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (n = 0, 1, 2, \dots).$$

Source: [1, ch. 4]

Exercise 2.3. Let $f(z) = \frac{\cos z}{(z^2 + 1)^2}$.

- Find the singularities of f .
- Show that the integral of f around $|z| = 3$ is the sum of its integrals around small circles centred at the singularities.
- Evaluate $\oint_{|z|=3} f(z) dz$.

Solution. Singularities. $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ is entire and $(z^2 + 1)^2$ is a polynomial, so f is analytic except where $(z^2 + 1)^2 = 0$, that is, at $z = \pm i$.

Decomposition. Let C_1 and C_2 be the positively oriented circles $|z - i| = \frac{1}{2}$ and $|z + i| = \frac{1}{2}$. They lie inside $|z| = 3$, their interiors are disjoint, and f is analytic everywhere else. By Theorem 1.3,

$$\oint_{|z|=3} f(z) dz = \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz.$$

Around i . Write $f(z) = g_1(z)/(z - i)^2$ with $g_1(z) = \cos z/(z + i)^2$, which is analytic on and inside C_1 . By Theorem 2.2 with $n = 1$, the integral is $2\pi i g_1'(i)$. Now

$$g_1'(z) = \frac{-(z + i) \sin z - 2 \cos z}{(z + i)^3},$$

and $\sin i = i \sinh 1$, $\cos i = \cosh 1$, so

$$g_1'(i) = \frac{-2i \cdot i \sinh 1 - 2 \cosh 1}{(2i)^3} = \frac{2(\sinh 1 - \cosh 1)}{-8i} = \frac{-2e^{-1}}{-8i} = \frac{1}{4ie}.$$

Hence $\oint_{C_1} f(z) dz = 2\pi i \cdot \frac{1}{4ie} = \frac{\pi}{2e}$.

Around $-i$. Likewise, with $g_2(z) = \cos z/(z - i)^2$,

$$g_2'(-i) = \frac{-(-2i)(-i \sinh 1) - 2 \cosh 1}{(-2i)^3} = \frac{2(\sinh 1 - \cosh 1)}{8i} = -\frac{1}{4ie},$$

so $\oint_{C_2} f(z) dz = -\frac{\pi}{2e}$.

Total.

$$\oint_{|z|=3} \frac{\cos z}{(z^2 + 1)^2} dz = \frac{\pi}{2e} - \frac{\pi}{2e} = 0. \quad \triangleleft$$

Remark 2.4. The answer 0 can be predicted without computing. f is even, and the substitution $z = -w$ maps the circle $|z| = 3$ onto itself with the same orientation, so $\oint f(z) dz = \oint f(-w)(-dw) = -\oint f(w) dw$.

Exercise 2.5. Let Γ be a simple closed positively oriented contour, and let f be analytic inside and on Γ . For z_0 not on Γ :

- evaluate $\oint_{\Gamma} \frac{f'(z)}{z - z_0} dz$ by Cauchy's integral formula;

- (b) evaluate $\oint_{\Gamma} \frac{f(z)}{(z-z_0)^2} dz$ by the formula for derivatives;
- (c) compare the two;
- (d) say what goes wrong when z_0 lies on Γ .

Solution. By Theorem 2.2, f' is analytic wherever f is.

z_0 inside Γ . Theorem 2.1 applied to f' , and Theorem 2.2 with $n = 1$ applied to f , give

$$\oint_{\Gamma} \frac{f'(z)}{z - z_0} dz = 2\pi i f'(z_0) = \oint_{\Gamma} \frac{f(z)}{(z - z_0)^2} dz.$$

z_0 outside Γ . Both integrands are analytic on a simply connected domain containing Γ and its inside but not z_0 , so both integrals are 0 by Theorem 1.2.

Comparison. In either case the two integrals are equal.

z_0 on Γ . Both integrands are unbounded near z_0 , so they are not continuous on Γ . Neither formula applies, and the integrals are not even defined as ordinary contour integrals; they would have to be treated as improper integrals. $\triangleleft$

Remark 2.6. The equality has a one-line explanation. Off z_0 ,

$$\frac{d}{dz} \frac{f(z)}{z - z_0} = \frac{f'(z)}{z - z_0} - \frac{f(z)}{(z - z_0)^2},$$

and the integral of a derivative around a closed contour is 0.

3 CAUCHY'S ESTIMATES AND LIOUVILLE'S THEOREM

THEOREM 3.1 (Cauchy's estimates). Let f be analytic inside and on the circle $C_R: |z - z_0| = R$, with $|f(z)| \leq M$ on C_R . Then

$$|f^{(n)}(z_0)| \leq \frac{n! M}{R^n} \quad (n = 0, 1, 2, \dots).$$

Source: [I, ch. 4]

PROOF. On C_R the integrand in Theorem 2.2 satisfies $|f(z)/(z - z_0)^{n+1}| \leq M/R^{n+1}$, and C_R has length $2\pi R$. By Theorem 1.1,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \cdot \frac{M}{R^{n+1}} \cdot 2\pi R = \frac{n! M}{R^n}. \quad \blacksquare$$

THEOREM 3.2 (Liouville's theorem). A bounded entire function is constant.

Source: [I, ch. 4]

PROOF. Suppose $|f| \leq M$ on $\mathbb{C}$. Fix z_0 . For every $R > 0$, Theorem 3.1 with $n = 1$ gives $|f'(z_0)| \leq M/R$. Letting $R \rightarrow \infty$ gives $f'(z_0) = 0$. So $f' \equiv 0$ on the connected set $\mathbb{C}$, and f is constant. $\blacksquare$

Exercise 3.3. Let f be entire with $A \leq \operatorname{Im} f(z) \leq B$ for all z . Show that e^{if} is bounded and entire, and deduce that f is constant.

PROOF. Write $f = u + iv$, so $if = -v + iu$ and

$$|e^{if}| = |e^{-v} e^{iu}| = e^{-v}.$$

Since $A \leq v \leq B$, this gives $e^{-B} \leq |e^{if}| \leq e^{-A}$, so e^{if} is bounded. It is entire because it is a composition of the entire functions if and $\exp$, with $\frac{d}{dz} e^{if} = if' e^{if}$ by the chain rule.

By **Theorem 3.2**, $e^{if} \equiv c$ for some constant c , and $c \neq 0$ because the exponential never vanishes. Differentiating, $if'(z)e^{if(z)} = 0$, so $f' \equiv 0$ and f is constant. ■

Exercise 3.4. Let f be entire with $f^{(7)}$ bounded on $\mathbb{C}$. What kind of function is f ? Generalise, and decide whether the same holds for real functions.

Solution. By **Theorem 2.2**, $f^{(7)}$ is entire, and it is bounded, so it is a constant c by **Theorem 3.2**. Hence $f^{(n)} \equiv 0$ for $n \geq 8$. By Taylor's theorem (**Theorem 8.2**) the series of f about 0 converges to f on all of $\mathbb{C}$, and its coefficients $f^{(n)}(0)/n!$ vanish for $n \geq 8$. So

$$f(z) = \sum_{n=0}^7 \frac{f^{(n)}(0)}{n!} z^n,$$

a polynomial of degree at most 7, and exactly 7 when $c \neq 0$.

Generalisation. The same argument shows that if f is entire and $f^{(k)}$ is bounded, then f is a polynomial of degree at most k .

Real functions. This fails on $\mathbb{R}$. For $g(x) = \sin x$, every derivative is $\pm \sin x$ or $\pm \cos x$, so $g^{(7)}(x) = -\cos x$ is bounded, yet $\sin x$ is not a polynomial. ◁

Remark 3.5. Applying **Theorem 3.1** directly to $g = f^{(7)}$ gives $|f^{(8)}(z_0)| \leq M/R$ for every R , which is the same argument that proves Liouville's theorem.

4 THE MAXIMUM MODULUS PRINCIPLE

THEOREM 4.1 (Maximum modulus principle). Let D be a bounded domain, and let f be continuous on the closure of D and analytic in D . Then $|f|$ attains its maximum over the closure on the boundary of D . If $|f|$ also attains it at a point of D , then f is constant.

Source: [1, ch. 4]

Exercise 4.2. Let $f(z) = (z - \frac{1}{2})(z - \frac{1}{4})$ on the closed unit disc. Explain why $\max_{|z| \leq 1} |f(z)|$ is attained on $|z| = 1$, compute $|f|$ there, and find the maximum.

Solution. Where f is a polynomial, so it is analytic on the open unit disc and continuous on its closure. By **Theorem 4.1**, the maximum of $|f|$ is attained on $|z| = 1$.

On the circle. Write $z = e^{i\theta}$ and $a = \cos \theta$. For real c , $|z - c|^2 = (z - c)(\bar{z} - c) = 1 - 2ca + c^2$, so

$$|f(z)|^2 = \left(\frac{5}{4} - a\right) \left(\frac{17}{16} - \frac{1}{2}a\right) = \frac{1}{2}a^2 - \frac{27}{16}a + \frac{85}{64}.$$

Maximum. The derivative in a is $a - \frac{27}{16} < 0$ on $[-1, 1]$, so $|f|^2$ is largest at $a = -1$, that is, at $z = -1$, where it equals $\frac{225}{64}$. Hence

$$\max_{|z| \leq 1} |f(z)| = \frac{15}{8}. \quad \triangleleft$$

Remark 4.3. The triangle inequality gets there faster: for $|z| \leq 1$, $|f(z)| \leq (1 + \frac{1}{2})(1 + \frac{1}{4}) = \frac{15}{8}$, with equality at $z = -1$.

CHAPTER 2: POWER AND TAYLOR SERIES

Every function analytic on a disc is the sum of its Taylor series there, and power series can be differentiated, multiplied and compared term by term. Each section states one of these facts and then puts it to work.

5 RADIUS OF CONVERGENCE

THEOREM 5.1 (Cauchy–Hadamard). Let $\sum a_n(z - z_0)^n$ be a power series and define $R \in [0, \infty]$ by

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n},$$

with $1/0 = \infty$ and $1/\infty = 0$. The series converges absolutely for $|z - z_0| < R$ and diverges for $|z - z_0| > R$. We call R the radius of convergence.

Source: [I, ch. 5]

THEOREM 5.2 (Ratio test). If $a_n \neq 0$ for all large n and $L = \lim_{n \rightarrow \infty} |a_{n+1}/a_n|$ exists in $[0, \infty]$, then the radius of convergence of $\sum a_n(z - z_0)^n$ is $R = 1/L$.

Source: [I, ch. 5]

6 TERMWISE DIFFERENTIATION

THEOREM 6.1 (Termwise differentiation). Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence $R > 0$. Then f is analytic on $|z - z_0| < R$, and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1},$$

where the differentiated series also has radius of convergence R .

Source: [I, ch. 5]

PROOF. Translating, we may take $z_0 = 0$.

Same radius. Since $n^{1/n} \rightarrow 1$,

$$\limsup_{n \rightarrow \infty} |n a_n|^{1/n} = \limsup_{n \rightarrow \infty} |a_n|^{1/n},$$

so $\sum n a_n z^{n-1}$ has radius R by **Theorem 5.1**. Similarly, $\sum n(n-1) |a_n| r^{n-2}$ converges for every $0 \leq r < R$.

The derivative. Fix $|z| < R$, choose r with $|z| < r < R$, and let $g(z) = \sum n a_n z^{n-1}$. For $|w| < r$ with $w \neq z$,

$$\frac{f(w) - f(z)}{w - z} - g(z) = \sum_{n=2}^{\infty} a_n \left(\frac{w^n - z^n}{w - z} - n z^{n-1} \right).$$

Expanding $\frac{w^n - z^n}{w - z} = \sum_{j=0}^{n-1} w^j z^{n-1-j}$, the bracket equals $\sum_{j=1}^{n-1} z^{n-1-j} (w^j - z^j)$. Since $|w^j - z^j| \leq j r^{j-1} |w - z|$, the bracket is at most

$$|w - z| r^{n-2} \sum_{j=1}^{n-1} j = |w - z| \frac{n(n-1)}{2} r^{n-2}$$

in modulus. Hence

$$\left| \frac{f(w) - f(z)}{w - z} - g(z) \right| \leq \frac{|w - z|}{2} \sum_{n=2}^{\infty} n(n-1) |a_n| r^{n-2} \rightarrow 0 \quad \text{as } w \rightarrow z.$$

So $f'(z) = g(z)$ exists at every point of the disc. That is, f is analytic there. ■

Exercise 6.2. Find the radius of convergence and a closed form for $f(z) = \sum_{n=1}^{\infty} n^3 z^n$. Use it to evaluate $\sum_{n=1}^{\infty} n^3/3^n$ and $\sum_{n=3}^{\infty} n^3/2^n$.

Solution. Radius. With $a_n = n^3$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(1 + \frac{1}{n} \right)^3 \rightarrow 1,$$

so $R = 1$ by [Theorem 5.2](#).

Closed form. Start from the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

Differentiating term by term ([Theorem 6.1](#)) and multiplying by z replaces each coefficient a_n by na_n without changing the disc. Doing this three times gives

$$\begin{aligned} \frac{z}{(1-z)^2} &= \sum_{n=1}^{\infty} n z^n, \\ \frac{z(1+z)}{(1-z)^3} &= \sum_{n=1}^{\infty} n^2 z^n, \\ \frac{z(1+4z+z^2)}{(1-z)^4} &= \sum_{n=1}^{\infty} n^3 z^n. \end{aligned}$$

For instance, the last line comes from the quotient rule:

$$\frac{d}{dz} \frac{z+z^2}{(1-z)^3} = \frac{(1+2z)(1-z) + 3(z+z^2)}{(1-z)^4} = \frac{1+4z+z^2}{(1-z)^4}.$$

The sums. Both points lie inside $|z| < 1$. At $z = 1/3$,

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n} = \frac{\frac{1}{3} \cdot \frac{22}{9}}{\left(\frac{2}{3}\right)^4} = \frac{22}{27} \cdot \frac{81}{16} = \frac{33}{8}.$$

At $z = 1/2$ the full sum is $\frac{\frac{1}{2} \cdot \frac{13}{4}}{\left(\frac{1}{2}\right)^4} = 26$, and removing the terms $n = 1, 2$, which are $\frac{1}{2}$ and 2, leaves

$$\sum_{n=3}^{\infty} \frac{n^3}{2^n} = 26 - \frac{5}{2} = \frac{47}{2}. \quad \triangleleft$$

Remark 6.3. The operator $\mathfrak{D} = z \frac{d}{dz}$ sends z^n to $n z^n$, so $f = \mathfrak{D}^3 \left(\frac{1}{1-z} \right)$. The same trick sums $\sum p(n) z^n$ for any polynomial p .

Exercise 6.4. Let

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{4^n (n!)^2}.$$

Show that f is entire and that $f''(z) + \frac{1}{z} f'(z) + f(z) = 0$ for $z \neq 0$.

PROOF. f is entire. Fix $z \neq 0$ and let c_n be the n th term. Then

$$\left| \frac{c_{n+1}}{c_n} \right| = \frac{|z|^2}{4(n+1)^2} \rightarrow 0,$$

so the series converges absolutely for every z . Its radius of convergence is ∞ , and f is entire by [Theorem 6.1](#).

Derivatives. By [Theorem 6.1](#), twice,

$$f'(z) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n z^{2n-1}}{4^n (n!)^2}, \quad f''(z) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1) z^{2n-2}}{4^n (n!)^2}.$$

Every term of f' contains a factor z , so

$$\frac{1}{z}f'(z) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n z^{2n-2}}{4^n (n!)^2}$$

is again a convergent power series. In fact it is entire, so the singularity of $f'(z)/z$ at the origin is only apparent.

The equation. Adding the last two series term by term,

$$f''(z) + \frac{1}{z}f'(z) = \sum_{n=1}^{\infty} \frac{(-1)^n 4n^2 z^{2n-2}}{4^n (n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-2}}{4^{n-1}((n-1)!)^2},$$

since $4^n = 4 \cdot 4^{n-1}$ and $(n!)^2 = n^2((n-1)!)^2$. Put $k = n - 1$:

$$f''(z) + \frac{1}{z}f'(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k}}{4^k (k!)^2} = -f(z). \quad \blacksquare$$

Remark 6.5. This f is the Bessel function J_0 . Multiplying through by z gives Bessel's equation of order zero, $zf'' + f' + zf = 0$, which then holds at $z = 0$ as well.

7 UNIQUENESS OF COEFFICIENTS

THEOREM 7.1 (*Uniqueness of power series*). If $\sum a_n(z - z_0)^n$ converges to $f(z)$ on some disc $|z - z_0| < r$, then

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n \geq 0).$$

In particular, two power series with the same sum near z_0 have the same coefficients.

Source: [I, ch. 5]

PROOF. Applying [Theorem 6.1](#) k times,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k}, \quad |z - z_0| < r.$$

Putting $z = z_0$ kills every term except $n = k$, so $f^{(k)}(z_0) = k! a_k$. If two series have the same sum f near z_0 , both sets of coefficients equal $f^{(n)}(z_0)/n!$. ■

Exercise 7.2. Let $f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots$. Show that $f(z) = z + f(z^2)$, and find every power series g that satisfies $g(z) = z + g(z^2)$ near the origin.

Solution. f satisfies the equation. The coefficients are 0 or 1, with infinitely many 1s, so the radius of convergence is 1. For $|z| < 1$,

$$z + f(z^2) = z + \sum_{n=0}^{\infty} z^{2^{n+1}} = z^{2^0} + \sum_{n=1}^{\infty} z^{2^n} = f(z).$$

All solutions. Suppose $g(z) = \sum a_n z^n$ converges near 0 and $g(z) = z + g(z^2)$. Then

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots = a_0 + z + a_1 z^2 + 0z^3 + a_2 z^4 + \cdots,$$

and by [Theorem 7.1](#)

$$a_0 \text{ is free,} \quad a_1 = 1, \quad a_{2k} = a_k \ (k \geq 1), \quad a_{2k+1} = 0 \ (k \geq 1).$$

Write $n \geq 1$ as $n = 2^m q$ with q odd. Halving m times gives $a_n = a_q$, which is 1 if $q = 1$ and 0 otherwise. So $a_n = 1$ exactly when n is a power of 2, and

$$g(z) = c + f(z), \quad c \in \mathbb{C}.$$

Conversely, each such g works, because the constant appears on both sides of $g(z) = z + g(z^2)$. The solutions are the translates of f by constants. $\triangleleft$

Remark 7.3. The gaps in the exponents 2^n make f impossible to continue past the unit circle: every point of $|z| = 1$ is a singularity. This is a *natural boundary*, proved as a separate graduate proof.

8 PRODUCTS AND TAYLOR'S THEOREM

THEOREM 8.1 (Cauchy product). If $f(z) = \sum a_n (z - z_0)^n$ and $g(z) = \sum b_n (z - z_0)^n$ both converge on $|z - z_0| < r$, then on the same disc

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad c_n = \sum_{k=0}^n a_k b_{n-k}.$$

Source: [I, ch. 5]

THEOREM 8.2 (Taylor's theorem). If f is analytic on $|z - z_0| < \rho$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

for every z in that disc. In particular, the radius of convergence of the Taylor series is at least the distance from z_0 to the nearest point where f fails to be analytic.

Source: [I, ch. 5]

Exercise 8.3. Find the first four nonzero terms of the Maclaurin series of $1/\cos z$. Justify each step.

Solution. A series exists. $\cos z$ is entire and $\cos 0 = 1 \neq 0$, so $1/\cos z$ is analytic near 0. By Taylor's theorem it has a Maclaurin series $\sum b_n z^n$.

Multiply out. Write $\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots$. Since $\cos z \cdot \frac{1}{\cos z} = 1$, [Theorem 8.1](#) gives

$$\left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \cdots\right)(b_0 + b_1 z + b_2 z^2 + \cdots) = 1 + 0z + 0z^2 + \cdots.$$

Compare coefficients. By [Theorem 7.1](#) the coefficients of each power of z agree on both sides:

$$\begin{array}{ll} z^0: & b_0 = 1, \\ z^2: & b_2 - \frac{1}{2}b_0 = 0, \quad b_2 = \frac{1}{2}, \\ z^4: & b_4 - \frac{1}{2}b_2 + \frac{1}{24}b_0 = 0, \quad b_4 = \frac{5}{24}, \\ z^6: & b_6 - \frac{1}{2}b_4 + \frac{1}{24}b_2 - \frac{1}{720}b_0 = 0, \quad b_6 = \frac{61}{720}. \end{array}$$

The odd equations read $b_1 = 0$, $b_3 = \frac{1}{2}b_1 = 0$, and so on, as they must, since $1/\cos z$ is even. Hence

$$\frac{1}{\cos z} = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \frac{61z^6}{720} + \cdots \quad \triangleleft$$

Remark 8.4. The zeros of $\cos z$ nearest the origin are $\pm\pi/2$, so this series converges exactly on $|z| < \pi/2$. The numerators 1, 1, 5, 61 are the Euler (secant) numbers.

Exercise 8.5. Expand $p(z) = z^4$ in powers of $z - 1$.

Solution. p is entire, so by **Theorem 8.2** its Taylor series about 1 converges to it everywhere. The derivatives at 1 are $p(1) = 1$, $p'(1) = 4$, $p''(1) = 12$, $p'''(1) = 24$, $p^{(4)}(1) = 24$, and $p^{(j)} \equiv 0$ for $j \geq 5$. Dividing by $j!$,

$$z^4 = 1 + 4(z-1) + 6(z-1)^2 + 4(z-1)^3 + (z-1)^4. \quad \triangleleft$$

Remark 8.6. The coefficients are the binomial coefficients $\binom{4}{j}$, as they must be: $z^4 = (1 + (z-1))^4$.

CHAPTER 3: SINGULARITIES, LAURENT SERIES AND RESIDUES

Near an isolated singularity a function is no longer a power series, but it is still a series in both positive and negative powers. The negative part classifies the singularity, and a single coefficient, the residue, is all that survives when we integrate around it.

9 LAURENT SERIES

THEOREM 9.1 (Laurent's theorem). *If f is analytic in the annulus $r < |z - z_0| < R$, then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

there, where both halves of the series converge absolutely in the annulus and uniformly on every closed subannulus. The coefficients are unique, and

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

for any positively oriented circle $C: |\zeta - z_0| = \rho$ with $r < \rho < R$.

Source: [I, ch. 5]

DEFINITION 9.2. Let f be analytic in $0 < |z - z_0| < R$, with Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ there. The isolated singularity z_0 is

- *removable* if $a_n = 0$ for every $n < 0$;
- a *pole of order m* if $a_{-m} \neq 0$ and $a_n = 0$ for every $n < -m$;
- *essential* if $a_n \neq 0$ for infinitely many $n < 0$.

The *residue* of f at z_0 is $\text{Res}(f; z_0) = a_{-1}$.

Exercise 9.3. Find the Laurent series of $f(z) = e^{1/(2z^2)}$ about 0, classify the singularity, and find $\text{Res}(f; 0)$.

Solution. The series $e^w = \sum_{j=0}^{\infty} w^j/j!$ converges for every w , because e^w is entire. Putting $w = 1/(2z^2)$ gives, for every $z \neq 0$,

$$e^{1/(2z^2)} = \sum_{j=0}^{\infty} \frac{1}{j! 2^j z^{2j}} = 1 + \frac{1}{2z^2} + \frac{1}{8z^4} + \frac{1}{48z^6} + \cdots.$$

By uniqueness of Laurent coefficients (**Theorem 9.1**), this is the Laurent series on $0 < |z| < \infty$. It has infinitely many negative powers, so 0 is an essential singularity. Only even powers occur, so $a_{-1} = 0$ and $\text{Res}(f; 0) = 0$. $\triangleleft$

Remark 9.4. A zero residue does not mean a mild singularity: here it sits beside an essential singularity. It only says that $\oint f = 0$ around small circles.

10 REMOVABLE SINGULARITIES

THEOREM 10.1 (Removable singularities). *Let f have an isolated singularity at z_0 . The following are equivalent:*

- (i) z_0 is a removable singularity;
- (ii) $\lim_{z \rightarrow z_0} f(z)$ exists and is finite;
- (iii) f is bounded in some punctured disc $0 < |z - z_0| < \delta$.

In that case, setting $f(z_0) = \lim_{z \rightarrow z_0} f(z)$ makes f analytic at z_0 .

Source: [I, ch. 5]

Exercise 10.2. Show that $f(z) = \frac{1 - \cos z}{z^2}$ has a removable singularity at 0, and find the value of $f(0)$ that makes f analytic there.

Solution. For every z , $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$, so for $z \neq 0$

$$f(z) = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots = \frac{1}{2} - \frac{z^2}{24} + \frac{z^4}{720} - \cdots.$$

The Laurent series has no negative powers, so the singularity is removable. Equivalently, $\lim_{z \rightarrow 0} f(z) = \frac{1}{2}$ (**Theorem 10.1**). Setting $f(0) = \frac{1}{2}$ makes f the sum of an everywhere convergent power series, so f is entire. $\triangleleft$

11 ZEROS AND POLES

THEOREM 11.1 (Zeros and poles). *Let $m \geq 1$.*

- (i) *f has a pole of order m at z_0 if and only if $f(z) = g(z)/(z - z_0)^m$ near z_0 , with g analytic at z_0 and $g(z_0) \neq 0$.*
- (ii) *g has a zero of order m at z_0 if and only if $g(z) = (z - z_0)^m h(z)$ with h analytic at z_0 and $h(z_0) \neq 0$.*
- (iii) *If g has a zero of order m at z_0 , then $1/g$ has a pole of order m there.*

Source: [I, ch. 5]

Exercise 11.2. Find the poles of $f(z) = \frac{e^z}{z^2(z-2)^2}$ and their orders.

Solution. e^z is entire, so f is analytic except at the zeros 0 and 2 of the denominator.

At 0, $f(z) = g(z)/z^2$ with $g(z) = e^z/(z-2)^2$, which is analytic at 0 with $g(0) = \frac{1}{4} \neq 0$. By [Theorem 11.1](#), 0 is a pole of order 2.

At 2, $f(z) = g(z)/(z-2)^2$ with $g(z) = e^z/z^2$, analytic at 2 with $g(2) = e^2/4 \neq 0$. So 2 is also a pole of order 2. $\triangleleft$

12 RESIDUES

THEOREM 12.1 (*Cauchy's residue theorem*). Let Γ be a simple closed positively oriented contour, and let f be analytic inside and on Γ except at finitely many points $z_1, \dots, z_n$ inside Γ . Then

$$\oint_{\Gamma} f(z) \, dz = 2\pi i \sum_{k=1}^n \text{Res}(f; z_k).$$

Source: [1, ch. 6]

PROOF. Surround each z_k by a small positively oriented circle C_k , with the discs disjoint and inside Γ . By [Theorem 1.3](#), $\oint_{\Gamma} f = \sum_k \oint_{C_k} f$. On C_k , f equals its Laurent series about z_k , which converges uniformly there ([Theorem 9.1](#)), so it may be integrated term by term. Since

$$\oint_{C_k} (z - z_k)^n \, dz = \begin{cases} 2\pi i, & n = -1, \\ 0, & n \neq -1, \end{cases}$$

only the a_{-1} term survives, and $\oint_{C_k} f = 2\pi i \text{Res}(f; z_k)$. $\blacksquare$

THEOREM 12.2 (*Residue at a pole*). If f has a pole of order m at z_0 , then

$$\text{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)].$$

For a simple pole, $\text{Res}(f; z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z)$.

Source: [1, ch. 6]

PROOF. Near z_0 ,

$$(z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{m-1} + \cdots,$$

a power series once its removable singularity at z_0 is filled in. By [Theorem 7.1](#), its $(m-1)$ st derivative at z_0 is $(m-1)!a_{-1}$, and this derivative is continuous, so it equals the limit. $\blacksquare$

Exercise 12.3. Let $f(z) = \frac{1}{z^2 \sin z}$. Find $\text{Res}(f; 0)$ in two ways, first by [Theorem 12.2](#) and then from the Laurent series, and use it to evaluate $\oint_{|z|=1} f(z) \, dz$.

Solution. Order of the pole. From the series for $\sin z$,

$$z^2 \sin z = z^3 b(z), \quad b(z) = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots,$$

with $b(0) = 1 \neq 0$. So $z^2 \sin z$ has a zero of order 3 at 0, and f has a pole of order 3 there ([Theorem 11.1](#)).

A series for $z/\sin z$. This is $1/h(z)$, which is analytic near 0. Write $h = 1 - c$ with $c = \frac{z^2}{6} - \frac{z^4}{120} + \dots$. For small z , $|c| < 1$, and the geometric series gives

$$\frac{z}{\sin z} = 1 + c + c^2 + \dots = 1 + \frac{z^2}{6} + \left(\frac{1}{36} - \frac{1}{120}\right)z^4 + \dots = 1 + \frac{z^2}{6} + \frac{7z^4}{360} + \dots$$

By the formula. With $m = 3$,

$$\text{Res}(f; 0) = \frac{1}{2!} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \left[\frac{z}{\sin z} \right] = \frac{1}{2} \lim_{z \rightarrow 0} \left(\frac{1}{3} + \frac{7z^2}{30} + \dots \right) = \frac{1}{6},$$

differentiating term by term (Theorem 6.1).

From the Laurent series. Dividing by z^3 ,

$$f(z) = \frac{1}{z^3} \cdot \frac{z}{\sin z} = \frac{1}{z^3} + \frac{1}{6z} + \frac{7z}{360} + \dots,$$

so again $\text{Res}(f; 0) = a_{-1} = \frac{1}{6}$.

The integral. The singularities of f are the zeros $z = k\pi$ of $z^2 \sin z$, and only $z = 0$ lies inside $|z| = 1$. By Theorem 12.1,

$$\oint_{|z|=1} \frac{dz}{z^2 \sin z} = 2\pi i \cdot \frac{1}{6} = \frac{\pi i}{3}. \quad \triangleleft$$

Exercise 12.4. Find the singularities and the residues there of

$$(a) f(z) = \frac{\sin z}{z(z+1)}, \quad (b) g(z) = \frac{\cos z}{z(z-1)^2}.$$

Solution. (a) At 0. Since $\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \dots$, $f(z) \rightarrow 1$ as $z \rightarrow 0$. The singularity is removable (Theorem 10.1), so there are no negative powers and $\text{Res}(f; 0) = 0$.

(a) At -1 . $f(z) = \frac{\sin z/z}{z+1}$, and $\frac{\sin(-1)}{-1} = \sin 1 \neq 0$, so -1 is a simple pole and

$$\text{Res}(f; -1) = \lim_{z \rightarrow -1} \frac{\sin z}{z} = \sin 1.$$

(b) At 0. $g(z) = \frac{\cos z/(z-1)^2}{z}$, with numerator $1 \neq 0$ at $z = 0$. This is a simple pole, and

$$\text{Res}(g; 0) = \lim_{z \rightarrow 0} \frac{\cos z}{(z-1)^2} = 1.$$

(b) At 1. $g(z) = \frac{\cos z/z}{(z-1)^2}$, with numerator $\cos 1 \neq 0$ at $z = 1$, so this is a pole of order 2. By Theorem 12.2 with $m = 2$,

$$\text{Res}(g; 1) = \lim_{z \rightarrow 1} \frac{d}{dz} \frac{\cos z}{z} = \lim_{z \rightarrow 1} \frac{-z \sin z - \cos z}{z^2} = -\sin 1 - \cos 1. \quad \triangleleft$$

CHAPTER 4: REAL INTEGRALS BY RESIDUES

Some real integrals that resist every calculus technique become routine once they are read as contour integrals. The simplest case is an integral of a rational function of $\cos \theta$ and $\sin \theta$ over a full period, which becomes an integral around the unit circle.

13 INTEGRALS OVER A PERIOD

THEOREM 13.1 (Trigonometric integrals). Let $R(x, y)$ be a rational function with no poles on the circle $x^2 + y^2 = 1$. Then

$$\int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta = \oint_{|z|=1} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \frac{dz}{iz},$$

and the right-hand side is $2\pi i$ times the sum of the residues of its integrand inside $|z| = 1$.

Source: [1, ch. 6]

PROOF. Parametrise the circle by $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$. Then $dz = iz d\theta$, $\cos \theta = \frac{1}{2}(z + z^{-1})$ and $\sin \theta = \frac{1}{2i}(z - z^{-1})$, which turns the contour integral into the real one. The integrand on the right is rational in z , so it has finitely many poles, and **Theorem 12.1** applies. ■

Exercise 13.2. Evaluate $\int_0^{2\pi} \frac{d\theta}{1 + \frac{1}{4} \cos \theta}$.

Solution. To the circle. By **Theorem 13.1**, with C the circle $|z| = 1$,

$$I = \oint_C \frac{1}{1 + \frac{1}{8}(z + z^{-1})} \frac{dz}{iz} = \oint_C \frac{8z}{z^2 + 8z + 1} \frac{dz}{iz} = \frac{8}{i} \oint_C \frac{dz}{z^2 + 8z + 1}.$$

Poles. The roots of $z^2 + 8z + 1$ are $z_{1,2} = -4 \pm \sqrt{15}$, both simple. Since $\sqrt{15} \approx 3.87$, $z_1 = -4 + \sqrt{15} \approx -0.13$ lies inside C and $z_2 = -4 - \sqrt{15} \approx -7.87$ lies outside.

Residue. By **Theorem 12.2**,

$$\text{Res}\left(\frac{1}{(z - z_1)(z - z_2)}; z_1\right) = \frac{1}{z_1 - z_2} = \frac{1}{2\sqrt{15}}.$$

Total.

$$I = \frac{8}{i} \cdot 2\pi i \cdot \frac{1}{2\sqrt{15}} = \frac{8\pi}{\sqrt{15}}. \quad \triangleleft$$

Exercise 13.3. Show that for real a with $0 < |a| < 1$,

$$\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2\pi}{\sqrt{1 - a^2}}.$$

Solution. As above,

$$\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2}{ia} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2}{a}z + 1}.$$

The roots $z_{1,2} = (-1 \pm \sqrt{1 - a^2})/a$ are real with product 1 and $|z_1| < |z_2|$, so exactly one of them, z_1 , lies inside the circle. Its residue is $1/(z_1 - z_2) = a/(2\sqrt{1 - a^2})$, and so

$$\int_0^{2\pi} \frac{d\theta}{1 + a \cos \theta} = \frac{2}{ia} \cdot 2\pi i \cdot \frac{a}{2\sqrt{1 - a^2}} = \frac{2\pi}{\sqrt{1 - a^2}}.$$

With $a = \frac{1}{4}$ this gives $2\pi/\sqrt{15/16} = 8\pi/\sqrt{15}$, agreeing with **Exercise 13.2**. ◀

Remark 13.4. As $a \rightarrow \pm 1$ the integral blows up: one pole moves onto the unit circle, where $1 + a \cos \theta$ vanishes.

CHAPTER 5: REAL AND COMPLEX DIFFERENTIABILITY

Complex differentiability looks like the real definition with x replaced by z , but it is far more rigid. Here are two places where the theories part ways.

14 ONE DERIVATIVE GIVES ALL OF THEM

On $\mathbb{R}$, a function can be differentiable once and no more. On $\mathbb{C}$, an analytic function has derivatives of every order, all analytic (Theorem 2.2).

Exercise 14.1. Show that $f(x) = x^{5/3}$ is differentiable on $\mathbb{R}$, but f' is not differentiable at 0.

Solution. For $x \neq 0$, $f'(x) = \frac{5}{3}x^{2/3}$. At 0,

$$f'(0) = \lim_{b \rightarrow 0} \frac{b^{5/3}}{b} = \lim_{b \rightarrow 0} b^{2/3} = 0,$$

which agrees with $\frac{5}{3}x^{2/3}$ at $x = 0$. But

$$\frac{f'(b) - f'(0)}{b} = \frac{5}{3}b^{-1/3}$$

is unbounded as $b \rightarrow 0$, so $f''(0)$ does not exist. ◁

Remark 14.2. No analytic function behaves like this: once f' exists on an open set, so do f'', f''' , and the rest.

15 NO MEAN VALUE THEOREM

For a real function differentiable on $[a, b]$, some c between a and b has $f'(c) = \frac{f(b) - f(a)}{b - a}$. The special case $f(a) = f(b)$ is Rolle's theorem. Neither survives in $\mathbb{C}$.

Exercise 15.1. Let $g(z) = z^3$ and $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Show that $g(1) = g(\omega)$, but g' does not vanish anywhere on the segment from 1 to ω .

Solution. $\omega = e^{2\pi i/3}$, so $g(\omega) = e^{2\pi i} = 1 = g(1)$. If the mean value theorem held, some c on the segment would have $g'(c) = 3c^2 = 0$, that is, $c = 0$. But the segment consists of the points $1 + t(\omega - 1)$, $0 \leq t \leq 1$, with imaginary part $\frac{\sqrt{3}}{2}t$, which is zero only at $t = 0$, the point 1. So 0 is not on the segment. ◁

Remark 15.2. The exponential gives another example: $e^0 = e^{2\pi i}$, yet e^z is never zero.

REFERENCES

- [1] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering and Science*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2003.