

# TERMWISE DIFFERENTIATION

Complex Analysis, Theorem 6.1

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**THEOREM 6.1 (Termwise differentiation).** Suppose  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  has radius of convergence  $R > 0$ . Then  $f$  is analytic on  $|z - z_0| < R$ , and

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1},$$

where the differentiated series also has radius of convergence  $R$ .

Source: [1, ch. 5]

*PROOF.* Translating, we may take  $z_0 = 0$ .

Same radius. Since  $n^{1/n} \rightarrow 1$ ,

$$\limsup_{n \rightarrow \infty} |n a_n|^{1/n} = \limsup_{n \rightarrow \infty} |a_n|^{1/n},$$

so  $\sum n a_n z^{n-1}$  has radius  $R$  by Theorem 5.1. Similarly,  $\sum n(n-1)|a_n|r^{n-2}$  converges for every  $0 \leq r < R$ .

*The derivative.* Fix  $|z| < R$ , choose  $r$  with  $|z| < r < R$ , and let  $g(z) = \sum n a_n z^{n-1}$ . For  $|w| < r$  with  $w \neq z$ ,

$$\frac{f(w) - f(z)}{w - z} - g(z) = \sum_{n=2}^{\infty} a_n \left( \frac{w^n - z^n}{w - z} - n z^{n-1} \right).$$

Expanding  $\frac{w^n - z^n}{w - z} = \sum_{j=0}^{n-1} w^j z^{n-1-j}$ , the bracket equals  $\sum_{j=1}^{n-1} z^{n-1-j} (w^j - z^j)$ . Since  $|w^j - z^j| \leq j r^{j-1} |w - z|$ , the bracket is at most

$$|w - z| r^{n-2} \sum_{j=1}^{n-1} j = |w - z| \frac{n(n-1)}{2} r^{n-2}$$

in modulus. Hence

$$\left| \frac{f(w) - f(z)}{w - z} - g(z) \right| \leq \frac{|w - z|}{2} \sum_{n=2}^{\infty} n(n-1) |a_n| r^{n-2} \longrightarrow 0 \quad \text{as } w \rightarrow z.$$

So  $f'(z) = g(z)$  exists at every point of the disc. That is,  $f$  is analytic there. ■

## REFERENCES

- [1] E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis with Applications to Engineering and Science*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2003.